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Standards for Smart Grid

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Abstract

The smart grid is a new advanced electrical network promoted by the government to address environmental sustainability, energy generation, distribution and consumption efficiency issues. As one of the enabling technologies, Advanced Metering Infrastructure refers to systems which measure, collect and analyze energy usage, and interact with advanced devices such as smart meters through various communication media either on request or on pre-defined schedules. This paper focuses on proposing the technical requirements imposed on the communications network for AMI. Then we examine each of the proposed AMI consumer application standards found in open literature based on these requirements. We will discuss the system engineering approach taken by NIST/EPRI to develop standards for smart grid and AMI and highlight outstanding security and interoperability issues concerned for deploying smart grid AMI. We compare the performances of consumer application standards in addressing these open issues. Finally, we conclude and propose our future work. Keywords-smart grid; AMI; IP-based

Keywords- **Immune pathology; artificial immune system; negative selection algorithm; immunodeficiency; system Efficiency**

I. INTRODUCTION- REQUIREMENTS FOR SMART GRID AMI COMMUNICATIONS NETWORK

The requirements for a communications network between the smart meters and a utility's MDM system will be under the umbrella of the requirements for communications infrastructure for the smart grid [7]. As mentioned in the previous section, this communications network is an IP-based network. Nevertheless, the structure of the IP protocol suite, with its OSI layers and features of the upper layer protocols, needs to take account of the specific requirements for the smart grid. We conclude the requirements for this IP-based communications network are as follows:

• Standards-based—the IP suite and upper layer protocols should be standards-based to ensure interoperability and to support diverse applications.
• Open—the open standards provide the widest possible range of devices that can be employed, and the development of new devices and entry by new vendors is encouraged.
• Interactive—interactive applications enable active participation by consumers in demand response.
• Interoperable—the communications network will be composed of segments using different networking technologies and protocols. It is vitally important that these segments can interoperate with each other to provide end-to-end services.
• Manageable—supports network management tools for network performance monitoring and management.
• Scalable—support for scaling to large deployments is a must for a smart grid communications network; 120 million IP nodes, in US alone, are required for the smart grid in the future.
• Extensible—extensibility facilitates support for new applications and services of smart grid.
• Upgradable—support for easy and gradual upgrade of different segments of the communications network is necessary to protect the investment in legacy components.
• Future-proof—enables new products, services, and markets; provides broad investment protection over the network's long lifetime.
• Resilient and Self-healing—high reliability and selfhealing requirements demand that the network be resilient and capable of continued operation, even in the presence of localized faults, and moreover be capable of self-healing from power disturbance events to minimize the effects of, and reduce recovery time from, network outages.
• Real-time—smart grid AMI applications, including demand response and dynamic pricing, require the communications network to provide metering data in near real-time.
• Cost-effective—the benefits of smart grid AMI load control and dynamic pricing programs could be outweighed by the increased cost of implementing a comprehensive smart grid AMI system. It is essential to implement a cost-effective smart grid AMI system.
• Supporting traffic differentiation—packets delivered on the smart grid AMI network are generated for different applications, such as more critical load control messages, and less delay-sensitive metering data. The AMI communications system needs to support traffic differentiation and prioritization in order to maximize the overall satisfaction for all purposes.

• Secure—the smart grid components have critical features, such as smart meters that accept utility commands to turn on and off, or in-home energy management systems and appliances that accept signals to turn down or off during peak energy demand times, all of which could be used in malicious ways. With the transmission of data through a public network, which hackers have easy access to for eavesdropping or tampering, ensuring an end-to-end secure transmission is a critical and challenging problem to tackle [8][9].

II. PROPOSED CONSUMER APPLICATION STANDARDS

In this section, we introduce several standards proposed by ANSI, IETF, and other standardization organizations, as AMI consumer applications. Most importantly, we analyze these standards against the above requirement metrics, and highlight the strengths and weaknesses of each standard.

A. ANSI C12.22

ANSI C12.22 is the American National Standards Institute standard that defines a set of application layer messaging services that are applicable for the enterprise and End Device components of an AMI for the smart grid. The messaging services are tailored for, but not limited to, the exchange of the data Table Elements defined and published in ANSI C12.19, IEEE P1377/D1, and MC1219. This standard uses AES encryption to enable strong, secure communications, including confidentiality and data integrity. Its security model is extensible to support new security mechanisms. Unlike C12.18 or C12.21 protocols, which only support session oriented communications, C12.22 provides both session and sessionless communications that help to reduce the complexity of handling communication links on both sides with less signaling overhead. Because of its independence from the underlying network technologies, the ANSI C12.22 open standard enables interoperability between AMI end devices and the utility data management system across heterogeneous network segments. ANSI C12.22 works with other layers of the IP protocol suite to achieve an end-to-end communication. The C12.22 IP communications system has the following key elements [5][10]:

• C12.22 IP Node—a C12.22 Node that is located on a C12.22 IP Network Segment and communicates using the IP protocol.

• C12.22 IP Network Segment—a collection of all C12.22 IP Nodes that implement the IP-based protocols, and can communicate with each other using IP routers, switches, and bridges and without the use of a C12.22 Relay.

• C12.22 IP Relay—a C12.22 IP Node that performs the functions of a C12.22 Relay. A C12.22 IP Relay acts as a bridge between a C12.22 IP Network Segment and an adjacent, C12.22 Network Segment. The C12.22 IP Relay maintains the routing information of devices under it.

• C12.22 IP Network—a C12.22 IP communications infrastructure composed of C12.22 IP Network Segments, interconnected using C12.22 IP Relays. It includes at least one C12.22 Master Relay.

• C12.22 IP Master Relay—a C12.22 IP Node located within the utility enterprise data center premises (head end). It operates at the top of the C12.22 Relay hierarchy and provides registration and re-registration services to all C12.22 devices in its domain. It contains all routing information to all accessible devices in the hierarchy and a list of notification hosts.

• C12.22 Communications Module—hardware that attaches a C12.22 Device to a C12.22 Network Segment. It can be physically located inside or outside the C12.22 device enclosure.

• C12.22 Device—a device which hosts a C12.22 application and interfaces to a C12.22 Communication Module.

• C12.22 IP Host—a C12.22 Node that contains a C12.22 application. Special IP Hosts are: C12.22 Authentication Host, which provides registration and de-registration of C12.22 Nodes in the C12.22 Master Relay domain; C12.22 Notification Host, which is notified and keeps track of activated and de-activated C12.22 Nodes in the network.

• C12.22 gateway—a device that is used to communicate between the C12.22 Nodes and non-C12.22 Nodes. It translates the ANSI C12.22 to and from other non-C12.22 protocols. We show the C12.22 IP communication

B. Open Standards

In the following subsections the open IEC standards for distributed control—IEC 61499—as well as for substation automation—IEC 61850—which are used for this work are explained in more detail.

1) IEC 61499 - Function Blocks: An approach to handle the increased complexity of the next generation of automation systems is provided by the IEC 61499 standard [6]. The IEC 61499 reference model has been developed especially as a methodology for modelling open distributed Industrial Process, Measurement and Control Systems (IPMCS) to obtain a vendor-independent system architecture. This new standard therefore serves as a reference architecture that is developed especially for distributed, modular, and flexible control systems and meets the fundamental requirements of open distributed systems as defined in [7] and [8]. The IEC 61499 standard has even more ambitious objectives. They can be described by examining the three issues portability, configurability, and interoperability (details are described in [5] and [9]):

• *Portability:* the ability of software tools to accept and interpret correctly library elements produced by other software tools.
• *Configurability:* the ability of (control) devices and their software components to be configured (i.e., selected, assigned locations, interconnected and parametrized) by multiple software tools.
• *Interoperability:* the ability of (control) devices from different vendors operating together to perform the functions
specified by one or more distributed applications.

The standard defines concepts and models that allow modular control software which is encapsulated in Function Blocks (FB) to be assembled to control applications and later on distributed to (embedded) controller nodes (i.e., called DEVICES in IEC 61499). It specifies an architectural model in a generic way and extends the FB model of its predecessor IEC 61131-3 [10] with an additional event handling mechanism. FBs are an established concept for industrial applications to define robust and reusable software components. They can store the software solution for various problems and they have a defined set of input and output parameters, which can be used to connect them to form complete automation and applications. One big difference between IEC 61499 and IEC 61131- 3 [10] is the execution model, as already mentioned above. IEC 61131-3 has a cyclic execution model for control algorithms but IEC 61499 is based on events, and this means that IEC 61499 also supports asynchronous execution [9]. As a result, distributed IEC 61499 control applications and/or application parts can be executed in a synchronous way through time triggered events but also in an asynchronous way [11]. The underlying communication network or field bus and its corresponding protocols are not directly in the scope of the IEC 61499 standard. A compliance profile for feasibility demonstrations, which was provided by the Holonic Manufacturing Systems (HMS) consortium, specifies its usage based on Ethernet [5] for the distributed IEC 61499 control applications. Nevertheless also other communication networks and field bus concepts can be easily integrated in an IEC 61499 solution.

In summary the most important concepts of IEC 61499 are an *event-driven execution model*, the possibility of *distributing control applications* to different control devices, a *management interface* capable of a basic reconfiguration support and an *application-centred modelling methodology*. The following main features characterize this relatively new standard [12]:

• Component oriented basic building blocks called FBs,
• Graphical intuitive way of modelling control algorithms done through connecting the in- and outputs of FBs,
• Direct support for distribution,
• Definitions for the interaction between devices of different vendors,
• Basic support for reconfiguration, and
• Based on existing standards of the domain (i.e., IEC 61131-3 compliant programming languages for the specification of datatypes and control algorithms, XML for the exchange of FB types and applications between
different software tools, etc.).

The IEC 61499 standard therefore provides an ideal starting base for the control architecture of next generation automation and control systems, especially also in the domain of "Smart Grids". Moreover closed loop control applications realized with IEC 61499 has already been discussed in [13], [14], and [15].

2) IEC 61850 - Power System Automation: The first edition of the IEC 61850 standard was published between 2003 and 2005 under the title "Communication networks and systems in substations" [16]. Since then some of the parts have been re-edit and the standard has also been extended to include automation outside substations. Thus the standard has been renamed to "Communication networks and systems for power utility automation". IEC 61850 has the intention to standardize the information used in power utility automation and to provide it in an object oriented matter that can simplify the interoperability between components from different vendors. The standard consists of ten different parts, but thematically the contents of IEC 61850 can be divided into three main parts: *modelling*, *configuration* and *communication*. The modelling aspect of IEC 61850 covers the mapping of physical system data into the data model used in IEC 61850 [17]. IEC 61850 uses so called Logical Nodes (LN) to model devices and functions of the power system, e.g. a circuit breaker, measurement or a voltage controller. The properties of the LNs are presented in a tabular format. In Figure 1 an example of the logical node for a non-phase related measurement (MMXN) is shown. In part 6 of IEC 61850 the System Configuration Language (SCL) is described [18]. SCL is based on XML and is intended to be a complete configuration tool with the means to configure

Applications of the IEC 61970 and IEC 61968 Common Information Model (CIM) have been expanding from its traditional usage in power system modeling and data exchange into the role of a standardized semantic model for the Smart Grid. The NIST Smart Grid Interoperability Road Map has identified the need for a semantically consistent framework on which to base the Smart Grid and has selected the CIM as a central element across many functional areas of the Smart Grid not traditionally addressed by the CIM. One such area relates to how CIM works with the IEC 61850 power system communications standard that has also become an important part of the NIST Road Map for both substation communications and as the basis for other Smart Grid oriented communications. This has made harmonization of CIM and IEC 61850 critically important to the NIST Smart Grid Road Map's goal of interoperability. This paper will provide a technical overview of the relationship between CIM and IEC 61850, where they complement each other, where they differ, and why they need to work together using a consistent set of semantics. The paper will then present the results of an EPRI sponsored project to formulate a harmonization plan for CIM and IEC 61850 that is being worked through the IEC technical committees. Smart Grid technology is recognized as a key component of the solution to challenges such as increasing electric demand, an aging utility infrastructure and workforce, and the environmental impact of greenhouse gases produced during electric generation. Integrated Smart Grid solutions combine advanced sensing technology, two-way high-speed communications using the utilities assets, 24/7 monitoring and enterprise analysis software and related services to provide location-specific, real-time actionable data as well as home energy management solutions to provide enhanced services for the end-users. As a result, these solutions increase the efficiency and reliability of the electric grid while reducing the environmental impact of electric usage benefiting utilities, their customers, and the environment. Smart Grid solutions, including Distribution Automation, Asset Management, Demand Side Management, Demand Response, Distributed Energy Management and Advanced Metering Infrastructure, allow utilities to identify and correct a number of specific system issues through a single integrated, robust, and scalable Smart Grid platform. Example case studies of these applications will be presented, analyzing the full solution deployment: hardware, telecomms, systems and applications, so as to demonstrate how the Smart Grid is enabling reduction of operational expenses, improvement of SAIFI and SAIDI, enhancing asset management, and improving distribution operations.

III. SMART GRID INTEROPERABILITY PANEL

The SGIP is a public-private partnership that is intended to be a permanent body that supports NIST in coordinating the framework of interoperable standards for the Smart grid. The SGIP will identify and address additional gaps, reflect changes in technology and requirements in the standards, as well as provide ongoing coordination of SSO efforts to support timely availability of new or revised SG standards. This body will align with the objective of NIST, which is to coordinate, not develop standards, and support their acceleration and harmonization. It will reviews the narrative use cases that describe the interactions and information exchanged among the SG applications, identify requirements to achieve SG functionality, and coordinate conformance testing. As of February, 2010, membership included over 500 member organizations and 1,400 individual representatives. 33 international companies/organizations from 12 different countries in the Americas, Europe and Asia are represented. The group has 22 stakeholder categories comprising utilities, renewable power suppliers, electric equipment suppliers, information communication technologies, appliance makers, automation suppliers, standards developers, regulators, and venture capital. The objective is to have an open, transparent process with international participation as well. The SGIP held its first meeting via teleconference in February, 2010 and will have several face-to-face meetings this year. As shown in Fig.3, the SGIP comprises a governing board with one member from each of the 22 stakeholder categories, two standing committee members, three at-large members, and exofficio members such as George Arnold, the NIST National Coordinator for SG Interoperability. The two permanent committees at present are the Architecture Committee and Testing and Certification Committee. A key to meeting the openness and transparency goals is the availability of all the SGIP activities, meetings, and outputs to all.

A. Next steps

The next steps in the NIST process are to create roadmaps for cyber security, testing and certification of SG devices and systems, and a roadmap for the Architecture Committee, including mapping of the PAPs to Conceptual Model. Because of the plethora of SG meetings, in order to minimize the travel demands on the SGIP members, face-to-face meetings of the SGIP will be co-located with other large SG meetings. The kickoff meeting is the exception, scheduled for March, but future meetings for the SGIP will be held in conjunction with other SG meetings in May and December. Once the SGIP produces results and recommendations, future releases of the NIST Interoperability Framework and Roadmap document will be made to reflect them. In addition, other SG issues not addressed by the Release 1.0 document such as electromagnetic interference (EMI) in SG systems, privacy of customer energy usage data, and safety will be included.

IV. SMART GRID STANDARDS

There are many applications, techniques and technological solutions for smart grid system that have been developed or are still in the development phase. However, the key challenge is that the overall smart grid system is lacking widely accepted standards and this situation prevents the integration of advanced applications, smart meters, smart devices and renewable energy sources and limits the inter-operability between them. The adoption of inter-operability standards for the overall system is a critical prerequisite for making the smart grid system a reality. Seamless interoperability, robust information security, increased safety of new products and systems, compact set of protocols and communication exchange are some of the objectives that can be achieved with smart grid standardization efforts [37]. There are many regional and national attempts towards achieving this goal; for example, the European Union Technology Platform organization's strategic energy technology plan is all about the development of a smart electricity system over the next 30 years; Ontario Energy Board, Canada, has committed itself towards the completion of a smart meter installation [37]. On the other hand, NIST, the American National Standards Institute (ANSI), the International Electro technical Commission (IEC), the Institute of Electrical and Electronics Engineers (IEEE), the International Organization for Standardization (ISO), the International Telecommunication Union (ITU), the 3rd Generation Partnership Project (3GPP) and on the regional level, the Korean Agency for Technology and Standards (KATS) and Joint In announced formation Systems Committee(JISC) are the recognized standard development organizations that are worth to mention. In addition, the CEN, CENELEC and ETSI has formed a Joint working group for smart grid standardization efforts and aim to achieve the European Commission's policy objectives regarding the smart grid [37]. Their efforts focus on smart metering functionalities and communication interfaces for electric, water and heat sectors in Europe. An overview of smart grid standards. In the following, the details of these standards are explained. A. Revenue Metering Information Model

ANSI C12.19: ANSI C12.19 is an ANSI standard for utility industry end device data tables. This standard is defining a table structure for data transmissions between an end device and a computer for utility applications using binary codes and XML content. ANSI C12.19 is not interested in defining device design criteria or specifying the language or protocol used to transport that data.

M-Bus: M-Bus is a European standard and provides the requirements for remotely reading all kinds of utility meters. The utility meters are connected to a common master that periodically reads the meters via M-Bus. The wireless version, Wireless M-Bus, is also specified recently.

ANSI C12.18: ANSI C12.18 is an American National Standard (ANSI) standard that is specifically designed for meter communications and responsible for two way communications between smart electricity meters (C12.18 device) and a C12.18 client via an optical port.

A. Building Automation

BACnet: BACnet is a standard communication protocol that was developed by the American Society of Heating, Refrigerating and Air-Conditioning Engineers (ASHRAE) for building automation and control networks and support the implementation of intelligent buildings with full integration of computer-based building automation and control systems from multiple manufacturers.

B. Substation Automation

IEC 61850: IEC 61850 is a flexible, open standard that defines the communication between devices in transmission, distribution and substation automation systems. To enable seamless data communications and information exchange between the overall distribution networks, it is aimed to increase the scope of IEC 6180 to whole electric network and provide its compatibility with Common Information Model (CIM) for monitoring, control and protection applications [25]. This technology is implemented by modern manufacturers in their latest power engineering products like distribution automation nodes/grid measurement and diagnostics devices [13].

C. Powerline Networking

HomePlug: HomePlug is a power line technology and the existing home electricity is used to connect the smart appliances to HAN; HomePlug Command and Control (HPCC) version is designed for low-cost applications. HomePlug is a promising technology to create a reliable HAN between electric appliances and a smart meter.

HomePlug Green PHY: HomePlug Green PHY specification is developed as a low power, cost-optimized power line networking specification standard for smart grid applications used in home area networking by the Smart Energy Technical Working Group within the HomePlug Powerline Alliance. The inputs for optimization of specifications for field tests were gathered from many utilities, i.e., Consumers Energy, Duke Energy, Pacific Gas and Electric, and Southern California Edison. Backwards interoperability, lower data rate and IP networking support, low power consumption, full interoperability with both HomePlug devices are the leading features of HomePlug Green PHY specification.

PRIME: PRIME is an open, global power line standard that provides multi-vendor interoperability and welcomes several entities to its body. Advanced Digital Design, CURRENT Group, Landis+Gyr, STMicroelectronics, uSyscom and ZIV Medida are some of the current companies that have extensive experience in PLC technology and smart metering.

G3-PLC: G3-PLC is a power line communications specification launched by ERDF and Maxim that aims to provide interoperability, cyber security and robustness and reduce infrastructure costs in smart grid implementations worldwide.

D. Home Area Network Device Communication Measurement and Control

U-SNAP: There have been a variety of incompatible standards for HAN. This lack of standardization in HAN Utility has driven major AMI suppliers and product manufacturers to develop a solution, namely Utility SmartNetwork Access Port (U-SNAP). The main requirement is the existence of an interface to connect any type of product to a HAN. U-SNAP basically enables the standardization of a connector and serial interface and identifies the hardware interface, physical dimensions, data transfer, message contents and protocol specifics for HAN devices to provide many communication protocols to connect HAN devices to smart meters.

IEEE P1901: The IEEE P1901Working Group (WG) under the sponsorship of the IEEE Communications Society developed the IEEE P1901 standard for high speed power line communications to meet in-home multimedia, utility and smart grid application requirements [36]. Access control and physical layer specifications for broadband over power line networks are analyzed in detail and the access system with cell structure is defined by the IEEE P1901Working Group [36]. The IEEE P1901 standard has an important effect on communications technology by integrating power line communications into wireless networks with extensive features, such as high-speed, walls-penetration, etc.

Z-Wave: Z-Wave is an alternative solution to ZigBee that handles the interference with 802.11/b/g since it operates in the 800MHz range [20]. Z-Wave is not an open standard and developed by The Z-Wave Alliance, an international consortium of manufacturers. The simple, modular and low-cost features make Z-Wave one of the leading wireless technologies in home automation. ZWave can be easily embedded to consumer electronic appliances, such as lighting, remote control, security systems that require low-bandwidth data operations. B. The promotion of standards at European level EDF R&D has heavily promoted the use of CIM since 2004. At European level, these efforts succeeded in 2009 with the adoption of CIM by ENTSO-E (European Network of Transmission System Operators for Electricity) for improving the reliability of the European network : CIM was chosen as the new data exchange format between European TSOs (Transmission System Operator), and also at the Market level. EDF R&D supports the migration of European TSOs to CIM by providing CIM converters to ENTSO-E. EDF R&D will continue this effort towards European system operators, as there will be more and more exchanges between TSOs and DSOs (Distribution System Operators).

As IEC-61850 is becoming essential in the energy communication domains, EDF R&D is involved in a technological monitoring process exploring the possibility to preempt the migration in long term and the application of new technologies. The inter-operability is being proved by the current ERDF Substation Automation System (SAS) stage deployment. EDF as a member of IUA (Innovation Utility Alliance) is active in promoting IEC-61850 standard in Europe. Another area is disseminating good practices through

European projects. We promoted IEC CIM usage in Address project (Active Demand related project due to the impact of Distributed Energy Resources on Distribution Networks). 61850, and its harmonization with CIM is also promoted through the use of Digi²tal platform in DERri project (Distributed Energy Resources Research Infrastructures, www.der-ri.net), the aim of which is to offer research infrastructures to study the impact of decentralized production on the Distribution network. Digi²tal emphasizes the expertise acquired about the CIM data extended to the requirements of decentralized production and new study functions developed: VoltVar Control, State Estimator, Network reconfiguration.

It will illustrate the standardization of CIM and 61850 models in the context of decentralized production resources and will reflect a priority work axis of TC57 in 2010 and 2011. MENOFIS is dedicated to promote advanced functions regarding metering, network operation and mobility. MENOFIS aims at illustrating the distribution of intelligence and communication systems, the use of standardised data exchange formats (CIM, IEC 61850, DLMS-COSEM). Data collected in each point of the platform will be transmitted to Digi²tal platform and made available for the use of various applications, including location on GIS displays. These technical developments around the business models will be closely linked to the progress of the work groups on standardisation in the context of implementation of the Smart Grid. Furthermore, Digi²tal will have to illustrate the problems related to the wholesale and retail electricity markets as the introduction of Smart Grid require lining up of the technical problems of the network with the constraints of the electricity markets. The perspective of pairing Digi²tal with the methods of scientific calculation to facilitate analysis of large volumes of data is also an aspect which is studied by EDF R&D. The development of automated metering management (AMM) will provide the DSO with a wealth of new data about LV distribution network (load curves, data related to quality of supply, …) which will have to be transmitted and stored for analysis purposes. The use of Open Source products is also a technical strategy we want to investigate. That is also the reason why we have a great concern regarding open source platform like Eclipse A. National Standard IEEE 1547 As for planning and design there are several rules and specifications in place. Summarizing these specifications from [9], [10], we find that DG cannot regulate the voltage level or cause the area level to go outside of the acceptable ranges as defined by standards.

In addition, paralleling or during the initial connection into the system the voltage is not allowed to fluctuate more than 5% in either direction when compared to the existing grid voltage level. Every DG must have a visible isolation device to break its connection to the grid. The DG must have an interconnecting system of protection able to operate with electromagnetic interference as well as current and voltage surges. For specific standard numbers see [9] for more detailed information. The system protection and reliability standards focus on the interconnecting system of protection not the actual DG. Looking at solar for example, one cannot directly connect solar panels to the home network and deliver power into the system. These specifications state that there must be a certified power electronic interface and control device between the solar panel and the customers' meter that allows the output of the generating device to meet certain specifications. This is also true for all other types of DG. For simplicity the specifications refer to the DG but it is the power electronic device that controls and protects the system. The grid refers to the utility controlled power distribution grid, while local grid or network refers to the customer controlled power grid. The Point of Common Coupling (PCC) is the point where the utility grid meets the local grid, this point it typically tied to the customers' meter. By the existing rules for protection and reliability, the DG cannot reclose into a de-energised grid. The DG must not supply energy to the local network until after the main utility network is reconnected to the local grid. Also the DG system must detect islanding in the grid and disconnect within 2 seconds. These are a few major areas where future standards governing smart grids must be revised to achieve the desirable attributes of the future grids such as "self healing". Again by existing rules, if there is a fault on the grid the DG must detect the fault and disconnect from the grid. The DG is not allowed to reconnect to the grid until a pre-specified finite time has elapsed during which both voltage and frequency has recovered to acceptable ranges. For example, the voltage must be within the acceptable range and the frequency must be between 59.3 and 60.5 Hz. [10], [11] The DG must be able to withstand 220% of the system rated voltage. The grounding scheme cannot disrupt the coordination of the Ground Fault Protection or cause overvoltage past the equipment rating. [9] Voltage and frequency levels must be monitored by the customer at the PCC when the following conditions are met. The protection system must be able to independently detect both phase and ground faults on the utility system, pass the anti-islanding test and be certified for this purpose. By the existing rules there are minimal interconnections requirements if the total capacity of all certified DG's at the PCC are less than 30kW. The monitored voltage and frequency must respond to abnormalities within specific times breaking the connection between the DG and the grid. Two things make electricity unique and a challenge for Smart grid:

1) Lack of flow control (Grid Management and control transformation is needed – i.e., communications)
2) Electricity storage requirements (static or dynamic storage and load optimization/power electronics –efficiency) Change either of these and the grid delivery system will be transformed. Smart grid design and operation can enable this to happen.

The Energy Independence and Security Act of 2007 identified the National Institute of Standards and Technology (NIST) as the organization to develop an interoperability framework of standards and protocols for the US smart grid. IEEE and several organizations were specifically named as key organizations to work with NIST on this framework and the subsequent development of standards. To do this, it is necessary to identify what standards and protocols exist today and the functions that are needed in the future smart grid. A number of the functions needed are described in the Act itself. While this particular effort may be US based, smart grids are being proposed and implemented in other countries and regions as well, and the development of international standards that could be used throughout the world is the goal. NIST has formed several Domain Expert Working Groups (DEWGs) to identify use cases, identify key relevant standards and standards gaps for inclusion in the future Smart Grid Standards Interoperability Roadmap. [2]

The current groups include
• Building-to-Grid (B2G)

- Industrial-to-Grid (I2G)
- Home-to-Grid (H2G)
- Transmission and Distribution (T&D)

Others are being considered e.g. Vehicle to Grid and cyber security. These groups meet at conferences such as Gridweek and Grid Interop as well as on teleconferences with discussions and contributions included on a Twiki for use by group members. IEEE-SA is a participant as well as other SDOs and representatives from industry. The current focus is on developing use cases and a taxonomy with a December 2008 progress report to Congress. IEEE smart grid related international standards and standards activities are sponsored by a number of IEEE societies and standards coordinating committees (SCCs) and the importance of coordination of this activity has been recognized by the IEEE Standards Association Standards Board.

We consider the following anycast field equations defined over an open bounded piece of network and /or feature space $\Omega \subset R^d$. They describe the dynamics of the mean anycast of each of p node populations.

$$\begin{cases} (\frac{d}{dt}+l_i)V_i(t,r) = \sum_{j=1}^{p}\int_{\Omega} J_{ij}(r,\bar{r})S[(V_j(t-\tau_{ij}(r,\bar{r}),\bar{r})-h_{|j})]d\bar{r} \\ \qquad +I_i^{ext}(r,t), \qquad t\geq 0, 1\leq i\leq p, \\ V_i(t,r)=\phi_i(t,r) \qquad t\in[-T,0] \end{cases} \quad (1)$$

We give an interpretation of the various parameters and functions that appear in (1), Ω is finite piece of nodes and/or feature space and is represented as an open bounded set of R^d. The vector r and $\bar{r}$ represent points in Ω. The function $S: R \to (0,1)$ is the normalized sigmoid function:

$$S(z)=\frac{1}{1+e^{-z}} \qquad (2)$$

It describes the relation between the input rate v_i of population i as a function of the packets potential, for example, $V_i = v_i = S[\sigma_i(V_i - h_i)]$. We note V the $p-$ dimensional vector $(V_1,...,V_p)$. The p function $\phi_i, i=1,...,p,$ represent the initial conditions, see below. We note ϕ the $p-$ dimensional vector $(\phi_1,...,\phi_p)$. The p function $I_i^{ext}, i=1,...,p,$ represent external factors from other network areas. We note I^{ext} the $p-$ dimensional vector $(I_1^{ext},...,I_p^{ext})$. The $p\times p$ matrix of functions $J=\{J_{ij}\}_{i,j=1,...,p}$ represents the connectivity between populations i and j, see below. The p real values $h_i, i=1,...,p,$ determine the threshold of activity for each population, that is, the value of the nodes potential corresponding to 50% of the maximal activity. The p real positive values $\sigma_i, i=1,...,p,$ determine the slopes of the sigmoids at the origin. Finally the p real positive values $l_i, i=1,...,p,$ determine the speed at which each anycast node potential decreases exponentially toward its real value. We also introduce the function $S: R^p \to R^p$, defined by $S(x)=[S(\sigma_1(x_1-h_1)),...,S(\sigma_p - h_p))]$, and the diagonal $p\times p$ matrix $L_0 = diag(l_1,...,l_p)$. Is the intrinsic dynamics of the population given by the linear response of data transfer. $(\frac{d}{dt}+l_i)$ is replaced by $(\frac{d}{dt}+l_i)^2$ to use the alpha function response. We use $(\frac{d}{dt}+l_i)$ for simplicity although our analysis applies to more general intrinsic dynamics. For the sake, of generality, the propagation delays are not assumed to be identical for all populations, hence they are described by a matrix $\tau(r,\bar{r})$ whose element $\tau_{ij}(r,\bar{r})$ is the propagation delay between population j at $\bar{r}$ and population i at r. The reason for this assumption is that it is still unclear from anycast if propagation delays are independent of the populations. We assume for technical reasons that τ is continuous, that is $\tau \in C^0(\bar{\Omega}^2, R_+^{p\times p})$. Moreover packet data indicate that τ is not a symmetric function i.e., $\tau_{ij}(r,\bar{r}) \neq \tau_{ij}(\bar{r},r)$, thus no assumption is made about this symmetry unless otherwise stated. In order to compute the righthand side of (1), we need to know the node potential factor V on interval $[-T,0]$. The value of T is obtained by considering the maximal delay:

$$\tau_m = \max_{i,j(r,\bar{r}\in\bar{\Omega}\times\bar{\Omega})} \tau_{i,j}(r,\bar{r}) \qquad (3)$$

Hence we choose $T=\tau_m$

E. *Mathematical Framework*

A convenient functional setting for the non-delayed packet field equations is to use the space $F=L^2(\Omega,R^p)$ which is a Hilbert space endowed with the usual inner product:

$$\langle V,U\rangle_F = \sum_{i=1}^{p}\int_{\Omega} V_i(r)U_i(r)dr \qquad (1)$$

To give a meaning to (1), we defined the history space $C = C^0([-\tau_m, 0], F)$ with $\|\phi\| = \sup_{t\in[-\tau_m,0]} \|\phi(t)\| F,$ which is the Banach phase space associated with equation (3). Using the notation $V_t(\theta) = V(t+\theta), \theta \in [-\tau_m, 0],$ we write (1) as

$$\begin{cases} \dot{V}(t) = -L_0 V(t) + L_1 S(V_t) + I^{ext}(t), \\ V_0 = \phi \in C, \end{cases} \qquad (2)$$

Where

$$\begin{cases} L_1 : C \to F, \\ \phi \to \int_\Omega J(.,\bar{r})\phi(\bar{r}, -\tau(.,\bar{r}))d\bar{r} \end{cases}$$

Is the linear continuous operator satisfying $\|L_1\| \le \|J\|_{L^2(\Omega^2, R^{p\times p})}$. Notice that most of the papers on this subject assume Ω infinite, hence requiring $\tau_m = \infty.$

Proposition 1.0 If the following assumptions are satisfied.

1. $J \in L^2(\Omega^2, R^{p\times p}),$
2. The external current $I^{ext} \in C^0(R, F),$
3. $\tau \in C^0(\overline{\Omega^2}, R_+^{p\times p}), \sup_{\overline{\Omega^2}} \tau \le \tau_m.$

Then for any $\phi \in C,$ there exists a unique solution $V \in C^1([0,\infty), F) \cap C^0([-\tau_m, \infty, F)$ to (3)

Notice that this result gives existence on R_+, finite-time explosion is impossible for this delayed differential equation. Nevertheless, a particular solution could grow indefinitely, we now prove that this cannot happen.

F. *Boundedness of Solutions*

A valid model of neural networks should only feature bounded packet node potentials.

Theorem 1.0 All the trajectories are ultimately bounded by the same constant R if $I \equiv \max_{t\in R^+} \|I^{ext}(t)\|_F < \infty.$

Proof :Let us defined $f : R \times C \to R^+$ as

$$f(t, V_t) \stackrel{def}{=} \left\langle -L_0 V_t(0) + L_1 S(V_t) + I^{ext}(t), V(t) \right\rangle_F = \frac{1}{2}\frac{d\|V\|_F^2}{dt}$$

We note $l = \min_{i=1,\dots p} l_i$

$$f(t, V_t) \le -l\|V(t)\|_F^2 + (\sqrt{p|\Omega|}\|J\|_F + I)\|V(t)\|_F$$

Thus, if

$$\|V(t)\|_F \ge 2\frac{\sqrt{p|\Omega|}.\|J\|_F + I}{l} \stackrel{def}{=} R, \ f(t, V_t) \le -\frac{lR^2}{2} \stackrel{def}{=} -\delta < 0$$

Let us show that the open route of F of center 0 and radius R, B_R, is stable under the dynamics of equation. We know that $V(t)$ is defined for all $t \ge 0s$ and that $f < 0$ on ∂B_R, the boundary of B_R. We consider three cases for the initial condition V_0. If $\|V_0\|_C < R$ and set $T = \sup\{t \mid \forall s \in [0,t], V(s) \in \overline{B_R}\}.$ Suppose that $T \in R,$ then $V(T)$ is defined and belongs to $\overline{B_R}$, the closure of B_R, because $\overline{B_R}$ is closed, in effect to ∂B_R, we also have $\frac{d}{dt}\|V\|_F^2 |_{t=T} = f(T, V_T) \le -\delta < 0$ because $V(T) \in \partial B_R$. Thus we deduce that for $\varepsilon > 0$ and small enough, $V(T+\varepsilon) \in \overline{B_R}$ which contradicts the definition of T. Thus $T \notin R$ and $\overline{B_R}$ is stable.

Because f<0 on $\partial B_R, V(0) \in \partial B_R$ implies that $\forall t > 0, V(t) \in B_R$. Finally we consider the case $V(0) \in C\overline{B_R}$. Suppose that $\forall t > 0, V(t) \notin \overline{B_R},$ then $\forall t > 0, \frac{d}{dt}\|V\|_F^2 \le -2\delta,$ thus $\|V(t)\|_F$ is monotonically decreasing and reaches the value of R in finite time when $V(t)$ reaches ∂B_R. This contradicts our assumption. Thus $\exists T > 0 \mid V(T) \in B_R.$

Proposition 1.1 : Let s and t be measured simple functions on X. for $E\varepsilon M$, define

$$\phi(E) = \int_E s\, d\mu \qquad (1)$$

Then ϕ is a measure on M.

$$\int_X (s+t)d\mu = \int_X s\, d\mu + \int_X t d\mu \qquad (2)$$

Proof : If s and if $E_1, E_2, ...$ are disjoint members of M whose union is E, the countable additivity of μ shows that

$$\phi(E) = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E) = \sum_{i=1}^{n} \alpha_i \sum_{r=1}^{\infty} \mu(A_i \cap E_r)$$
$$= \sum_{r=1}^{\infty} \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E_r) = \sum_{r=1}^{\infty} \phi(E_r)$$

Also, $\varphi(\phi) = 0$, so that φ is not identically ∞. Next, let s be as before, let $\beta_1, ..., \beta_m$ be the distinct values of t,and let $B_j = \{x : t(x) = \beta_j\}$ If $E_{ij} = A_i \cap B_j$, the $\int_{E_{ij}} (s+t) d\mu = (\alpha_i + \beta_j)\mu(E_{ij})$

and $\int_{E_{ij}} s d\mu + \int_{E_{ij}} t d\mu = \alpha_i \mu(E_{ij}) + \beta_j \mu(E_{ij})$

Thus (2) holds with E_{ij} in place of X . Since X is the disjoint union of the sets E_{ij} $(1 \le i \le n, 1 \le j \le m)$, the first half of our proposition implies that (2) holds.

Theorem 1.1: If K is a compact set in the plane whose complement is connected, if f is a continuous complex function on K which is holomorphic in the interior of , and if $\varepsilon > 0$, then there exists a polynomial P such that $|f(z) = P(z)| < \varepsilon$ for all $z \varepsilon K$. If the interior of K is empty, then part of the hypothesis is vacuously satisfied, and the conclusion holds for every $f \varepsilon C(K)$. Note that K need to be connected.

Proof: By Tietze's theorem, f can be extended to a continuous function in the plane, with compact support. We fix one such extension and denote it again by f . For any $\delta > 0$, let $\omega(\delta)$ be the supremum of the numbers $|f(z_2) - f(z_1)|$ Where z_1 and z_2 are subject to the condition $|z_2 - z_1| \le \delta$. Since f is uniformly continous, we have $\lim_{\delta \to 0} \omega(\delta) = 0$ (1) From now on, δ will be fixed. We shall prove that there is a polynomial P such that

$$|f(z) - P(z)| < 10{,}000\ \omega(\delta) \quad (z \varepsilon K) \qquad (2)$$

By (1), this proves the theorem. Our first objective is the construction of a function $\Phi \varepsilon C_c^{'}(R^2)$, such that for all z

$$|f(z) - \Phi(z)| \le \omega(\delta), \qquad (3)$$

$$|(\partial \Phi)(z)| < \frac{2\omega(\delta)}{\delta}, \qquad (4)$$

And

$$\Phi(z) = -\frac{1}{\pi} \iint_X \frac{(\partial\Phi)(\zeta)}{\zeta - z} d\zeta d\eta \qquad (\zeta = \xi + i\eta), \qquad (5)$$

Where X is the set of all points in the support of Φ whose distance from the complement of K does not δ . (Thus X contains no point which is "far within" K.) We construct Φ as the convolution of f with a smoothing function A. Put $a(r) = 0$ if $r > \delta$, put

$$a(r) = \frac{3}{\pi\delta^2}(1 - \frac{r^2}{\delta^2})^2 \qquad (0 \le r \le \delta), \qquad (6)$$

And define

$$A(z) = a(|z|) \qquad (7)$$

For all complex z . It is clear that $A \varepsilon C_c^{'}(R^2)$. We claim that

$$\iint_{R^s} A = 1, \qquad (8)$$

$$\iint_{R^2} \partial A = 0, \qquad (9)$$

$$\iint_{R^3} |\partial A| = \frac{24}{15\delta} < \frac{2}{\delta}, \qquad (10)$$

The constants are so adjusted in (6) that (8) holds. (Compute the integral in polar coordinates), (9) holds simply because A has compact support. To compute (10), express ∂A in polar coordinates, and note that $\partial A / \partial \theta = 0$,

$$\partial A / \partial r = -a^{'},$$

Now define

$$\Phi(z) = \iint_{R^2} f(z - \zeta) A d\xi d\eta = \iint_{R^2} A(z - \zeta) f(\zeta) d\xi d\eta \qquad (11)$$

Since f and A have compact support, so does Φ. Since

$$\Phi(z) - f(z)$$
$$= \iint_{R^2} [f(z - \zeta) - f(z)] A(\xi) d\xi d\eta \quad (12)$$

And $A(\zeta) = 0$ if $|\zeta| > \delta$, (3) follows from (8). The difference quotients of A converge boundedly to the corresponding partial derivatives, since

$A\varepsilon C_c^{'}(R^2)$. Hence the last expression in (11) may be differentiated under the integral sign, and we obtain

$$(\partial\Phi)(z)=\iint_{R^2}(\partial\overline{A})(z-\zeta)f(\zeta)d\xi d\eta$$
$$=\iint_{R^2}f(z-\zeta)(\partial A)(\zeta)d\xi d\eta$$
$$=\iint_{R^2}[f(z-\zeta)-f(z)](\partial A)(\zeta)d\xi d\eta \qquad (13)$$

The last equality depends on (9). Now (10) and (13) give (4). If we write (13) with Φ_x and Φ_y in place of $\partial\Phi$, we see that Φ has continuous partial derivatives, if we can show that $\partial\Phi=0$ in G, where G is the set of all $z\varepsilon K$ whose distance from the complement of K exceeds δ. We shall do this by showing that

$$\Phi(z)=f(z) \qquad (z\varepsilon G); \qquad (14)$$

Note that $\partial f=0$ in G, since f is holomorphic there. Now if $z\varepsilon G$, then $z-\zeta$ is in the interior of K for all ζ with $|\zeta|<\delta$. The mean value property for harmonic functions therefore gives, by the first equation in (11),

$$\Phi(z)=\int_0^{\delta}a(r)rdr\int_0^{2\pi}f(z-re^{i\theta})d\theta$$
$$=2\pi f(z)\int_0^{\delta}a(r)rdr=f(z)\iint_{R^2}A=f(z) \qquad (15)$$

For all $z\ \varepsilon\ G$, we have now proved (3), (4), and (5) The definition of X shows that X is compact and that X can be covered by finitely many open discs $D_1,\dots,D_n$, of radius 2δ, whose centers are not in K. Since S^2-K is connected, the center of each D_j can be joined to ∞ by a polygonal path in S^2-K. It follows that each D_j contains a compact connected set E_j, of diameter at least 2δ, so that S^2-E_j is connected and so that $K\cap E_j=\phi.$ with $r=2\delta$. There are functions $g_j\varepsilon H(S^2-E_j)$ and constants b_j so that the inequalities.

$$|Q_j(\zeta,z)|<\frac{50}{\delta}, \qquad (16)$$

$$\left|Q_j(\zeta,z)-\frac{1}{z-\zeta}\right|<\frac{4{,}000\delta^2}{|z-\zeta|^2} \qquad (17)$$

Hold for $z\notin E_j$ and $\zeta\in D_j$, if

$$Q_j(\zeta,z)=g_j(z)+(\zeta-b_j)g_j^2(z) \qquad (18)$$

Let Ω be the complement of $E_1\cup\dots\cup E_n$. Then Ω is an open set which contains K. Put $X_1=X\cap D_1$ and $X_j=(X\cap D_j)-(X_1\cup\dots\cup X_{j-1}),$ for $2\le j\le n,$

Define

$$R(\zeta,z)=Q_j(\zeta,z) \qquad (\zeta\varepsilon X_j, z\,\varepsilon\,\Omega) \qquad (19)$$

And

$$F(z)=\frac{1}{\pi}\iint_X(\partial\Phi)(\zeta)R(\zeta,z)d\zeta d\eta \qquad (20)$$
$$(z\ \varepsilon\ \Omega)$$

Since,

$$F(z)=\sum_{j=1}\frac{1}{\pi}\iint_{X_i}(\partial\Phi)(\zeta)Q_j(\zeta,z)d\xi d\eta, \qquad (21)$$

(18) shows that F is a finite linear combination of the functions g_j and g_j^2. Hence $F\varepsilon H(\Omega)$. By (20), (4), and (5) we have

$$|F(z)-\Phi(z)|<\frac{2\omega(\delta)}{\pi\delta}\iint_X|R(\zeta,z)$$
$$-\frac{1}{z-\zeta}|d\xi d\eta \quad (z\ \varepsilon\ \Omega) \quad (22)$$

Observe that the inequalities (16) and (17) are valid with R in place of Q_j if $\zeta\ \varepsilon\ X$ and $z\ \varepsilon\ \Omega$. Now fix $z\ \varepsilon\ \Omega$., put $\zeta=z+\rho e^{i\theta}$, and estimate the integrand in (22) by (16) if $\rho<4\delta$, by (17) if $4\delta\le\rho$. The integral in (22) is then seen to be less than the sum of

$$2\pi\int_0^{4\delta}\left(\frac{50}{\delta}+\frac{1}{\rho}\right)\rho d\rho=808\pi\delta \qquad (23)$$

And

$$2\pi\int_{4\delta}^{\infty}\frac{4{,}000\delta^2}{\rho^2}\rho d\rho=2{,}000\pi\delta. \qquad (24)$$

Hence (22) yields

$$|F(z)-\Phi(z)|<6{,}000\omega(\delta) \qquad (z\ \varepsilon\ \Omega) \qquad (25)$$

Since $F\ \varepsilon\ H(\Omega)$, $K\subset\Omega$, and S^2-K is connected, Runge's theorem shows that F can be uniformly approximated on K by polynomials. Hence (3) and (25) show that (2) can be satisfied. This completes the proof.

Lemma 1.0 : Suppose $f \varepsilon C_c^{'}(R^2)$, the space of all continuously differentiable functions in the plane, with compact support. Put

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \qquad (1)$$

Then the following "Cauchy formula" holds:

$$f(z) = -\frac{1}{\pi}\iint_{R^2}\frac{(\partial f)(\zeta)}{\zeta - z}d\xi d\eta$$

$$(\zeta = \xi + i\eta) \qquad (2)$$

Proof: This may be deduced from Green's theorem. However, here is a simple direct proof:

Put $\varphi(r,\theta) = f(z + re^{i\theta}),\ r > 0,\ \theta$ real

If $\zeta = z + re^{i\theta}$, the chain rule gives

$$(\partial f)(\zeta) = \frac{1}{2}e^{i\theta}\left[\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right]\varphi(r,\theta) \qquad (3)$$

The right side of (2) is therefore equal to the limit, as $\varepsilon \to 0$, of

$$-\frac{1}{2}\int_{\varepsilon}^{\infty}\int_{0}^{2\pi}\left(\frac{\partial \varphi}{\partial r} + \frac{i}{r}\frac{\partial \varphi}{\partial \theta}\right)d\theta dr \qquad (4)$$

For each $r > 0, \varphi$ is periodic in θ, with period 2π. The integral of $\partial\varphi / \partial\theta$ is therefore 0, and (4) becomes

$$-\frac{1}{2\pi}\int_{0}^{2\pi}d\theta\int_{\varepsilon}^{\infty}\frac{\partial \varphi}{\partial r}dr = \frac{1}{2\pi}\int_{0}^{2\pi}\varphi(\varepsilon,\theta)d\theta \qquad (5)$$

As $\varepsilon \to 0,\ \varphi(\varepsilon,\theta) \to f(z)$ uniformly. This gives (2)

If $X^{\alpha} \in a$ and $X^{\beta} \in k[X_1,...X_n]$, then $X^{\alpha}X^{\beta} = X^{\alpha+\beta} \in a$, and so A satisfies the condition $(*)$. Conversely,

$$(\sum_{\alpha\in A} c_{\alpha}X^{\alpha})(\sum_{\beta\in\square^n} d_{\beta}X^{\beta}) = \sum_{\alpha,\beta} c_{\alpha}d_{\beta}X^{\alpha+\beta} \qquad (\textit{finite sums}),$$

and so if A satisfies $(*)$, then the subspace generated by the monomials $X^{\alpha}, \alpha \in a$, is an ideal. The proposition gives a classification of the monomial ideals in $k[X_1,...X_n]$: they are in one to one correspondence with the subsets A of $\square^n$ satisfying $(*)$. For example, the monomial ideals in $k[X]$ are exactly the ideals $(X^n),\ n \geq 1$, and the zero ideal (corresponding to the empty set A). We write $\langle X^{\alpha} \mid \alpha \in A\rangle$ for the ideal corresponding to A (subspace generated by the $X^{\alpha}, \alpha \in a$).

LEMMA 1.1. Let S be a subset of $\square^n$. The the ideal a generated by $X^{\alpha}, \alpha \in S$ is the monomial ideal corresponding to

$$A \overset{df}{=} \left\{\beta \in \square^n \mid \beta - \alpha \in \square^n, \quad some\ \alpha \in S\right\}$$

Thus, a monomial is in a if and only if it is divisible by one of the $X^{\alpha}, \alpha \in| S$

PROOF. Clearly A satisfies $(*)$, and $a \subset \langle X^{\beta} \mid \beta \in A\rangle$. Conversely, if $\beta \in A$, then $\beta - \alpha \in \square^n$ for some $\alpha \in S$, and $X^{\beta} = X^{\alpha}X^{\beta-\alpha} \in a$. The last statement follows from the fact that $X^{\alpha} \mid X^{\beta} \Leftrightarrow \beta - \alpha \in \square^n$. Let $A \subset \square^n$ satisfy $(*)$. From the geometry of A, it is clear that there is a finite set of elements $S = \{\alpha_1,...\alpha_s\}$ of A such that $A = \left\{\beta \in \square^n \mid \beta - \alpha_i \in \square^2,\ some\ \alpha_i \in S\right\}$ (The $\alpha_i's$ are the corners of A) Moreover, $a \overset{df}{=} \langle X^{\alpha} \mid \alpha \in A\rangle$ is generated by the monomials $X^{\alpha_i}, \alpha_i \in S$.

DEFINITION 1.0. For a nonzero ideal a in $k[X_1,...,X_n]$, we let $(LT(a))$ be the ideal generated by

$$\{LT(f) \mid f \in a\}$$

LEMMA 1.2 Let a be a nonzero ideal in $k[X_1,...,X_n]$; then $(LT(a))$ is a monomial ideal, and it equals $(LT(g_1),...,LT(g_n))$ for some $g_1,...,g_n \in a$.

PROOF. Since $(LT(a))$ can also be described as the ideal generated by the leading monomials (rather than the leading terms) of elements of a.

THEOREM 1.2. Every *ideal* a in $k[X_1,...,X_n]$ is finitely generated; more precisely, $a = (g_1,...,g_s)$ where $g_1,...,g_s$ are any elements of a whose leading terms generate $LT(a)$

PROOF. Let $f \in a$. On applying the division algorithm, we find $f = a_1 g_1 + ... + a_s g_s + r, \quad a_i, r \in k[X_1,..., X_n]$, where either $r = 0$ or no monomial occurring in it is divisible by any $LT(g_i)$. But $r = f - \sum a_i g_i \in a$, and therefore $LT(r) \in LT(a) = (LT(g_1),..., LT(g_s))$, implies that every monomial occurring in r is divisible by one in $LT(g_i)$. Thus $r = 0$, and $g \in (g_1,..., g_s)$.

DEFINITION 1.1. A finite subset $S = \{g_1,| ..., g_s\}$ of an ideal a is a standard ($(Gr\ddot{o}bner)$ bases for a if $(LT(g_1),..., LT(g_s)) = LT(a)$. In other words, S is a standard basis if the leading term of every element of a is divisible by at least one of the leading terms of the g_i.

THEOREM 1.3 *The ring* $k[X_1,..., X_n]$ *is Noetherian i.e., every ideal is finitely generated.*

PROOF. For $n = 1$, $k[X]$ is a principal ideal domain, which means that every ideal is generated by single element. We shall prove the theorem by induction on n. Note that the obvious map $k[X_1,...X_{n-1}][X_n] \to k[X_1,...X_n]$ is an isomorphism – this simply says that every polynomial f in n variables $X_1,...X_n$ can be expressed uniquely as a polynomial in X_n with coefficients in $k[X_1,..., X_n]$:

$$f(X_1,...X_n) = a_0(X_1,...X_{n-1})X_n^r + ... + a_r(X_1,...X_{n-1})$$

Thus the next lemma will complete the proof

LEMMA 1.3. If A is Noetherian, then so also is $A[X]$

PROOF. For a polynomial

$$f(X) = a_0 X^r + a_1 X^{r-1} + ... + a_r, \quad a_i \in A, \quad a_0 \neq 0,$$

r is called the degree of f, and a_0 is its leading coefficient. We call 0 the leading coefficient of the polynomial 0. Let a be an ideal in $A[X]$. The leading coefficients of the polynomials in a form an ideal a' in A, and since A is Noetherian, a' will be finitely generated. Let $g_1,..., g_m$ be elements of a whose leading coefficients generate a', and let r be the maximum degree of g_i. Now let $f \in a$, and suppose f has degree $s > r$, say, $f = aX^s + ...$ Then $a \in a'$, and so we can write

$$a = \sum b_i a_i, \quad b_i \in A,$$

a_i = *leading coefficient of* g_i

Now

$f - \sum b_i g_i X^{s - r_i}$, $r_i = \deg(g_i)$, has degree $< \deg(f)$. By continuing in this way, we find that $f \equiv f_t \mod(g_1,...g_m)$ With f_t a polynomial of degree $t < r$. For each $d < r$, let a_d be the subset of A consisting of 0 and the leading coefficients of all polynomials in a of degree d; it is again an ideal in A. Let $g_{d,1},..., g_{d,m_d}$ be polynomials of degree d whose leading coefficients generate a_d. Then the same argument as above shows that any polynomial f_d in a of degree d can be written $f_d \equiv f_{d-1} \mod(g_{d,1},...g_{d,m_d})$ With f_{d-1} of degree $\leq d-1$. On applying this remark repeatedly we find $f_t \in (g_{r-1,1},...g_{r-1,m_{r-1}},...g_{0,1},...g_{0,m_0})$ Hence

$$f_t \in (g_1,...g_m g_{r-1,1},...g_{r-1,m_{r-1}},..., g_{0,1},..., g_{0,m_0})$$

and so the polynomials $g_1,..., g_{0,m_0}$ generate a

One of the great successes of category theory in computer science has been the development of a "unified theory" of the constructions underlying denotational semantics. In the untyped λ-calculus, any term may appear in the function position of an application. This means that a model D of the λ-calculus must have the property that given a term t whose interpretation is $d \in D$, Also, the interpretation of a functional abstraction like $\lambda x . x$ is most conveniently defined as a function from $D to D$, which must then be regarded as an element of D. Let $\psi : [D \to D] \to D$ be the function that picks out elements of D to represent elements of $[D \to D]$ and $\phi : D \to [D \to D]$ be the function that maps elements of D to functions of D. Since $\psi(f)$ is intended to represent the function f as an element of *D*, it makes sense to require that $\phi(\psi(f)) = f$, that is, $\psi o \psi = id_{[D\to D]}$ Furthermore, we often want to view every element of *D* as representing some function from *D to D* and require that elements representing the same function be equal – that is

$$\psi(\varphi(d)) = d$$

or

$$\psi o \phi = id_D$$

The latter condition is called extensionality. These conditions together imply that $\phi\, and\, \psi$ are inverses--- that is, *D* is isomorphic to the space of functions from *D to D* that can be the interpretations of functional abstractions: $D \cong [D \to D]$ .Let us suppose we are working with the untyped $\lambda - calculus$, we need a solution ot the equation $D \cong A + [D \to D]$, where A is some predetermined domain containing interpretations for elements of *C*. Each element of *D* corresponds to either an element of *A* or an element of $[D \to D]$, with a tag. This equation can be solved by finding least fixed points of the function $F(X) = A + [X \to X]$ from domains to domains --- that is, finding domains *X* such that $X \cong A + [X \to X]$, and such that for any domain *Y* also satisfying this equation, there is an embedding of *X* to *Y* --- a pair of maps

$$X \underset{f^R}{\overset{f}{\rightleftarrows}} Y$$

Such that

$$f^R o f = id_X$$

$$f o f^R \subseteq id_Y$$

Where $f \subseteq g$ means that $f\ approximates\ g$ in some ordering representing their information content. The key shift of perspective from the domain-theoretic to the more general category-theoretic approach lies in considering *F* not as a function on domains, but as a *functor* on a category of domains. Instead of a least fixed point of the function, *F*.

Definition 1.3: Let ***K*** be a category and $F: K \to K$ as a functor. A fixed point of *F* is a pair (A,a), where A is a ***K-object*** and $a: F(A) \to A$ is an isomorphism. A prefixed point of F is a pair (A,a), where A is a ***K-object*** and a is any arrow from F(A) to A

Definition 1.4 : An $\omega - chain$ in a category ***K*** is a diagram of the following form:

$$\Delta = D_o \xrightarrow{f_o} D_1 \xrightarrow{f_1} D_2 \xrightarrow{f_2} \ldots..$$

Recall that a cocone μ of an $\omega - chain$ Δ is a *K*-object *X* and a collection of K *–arrows* $\{\mu_i : D_i \to X \mid i \geq 0\}$ such that $\mu_i = \mu_{i+1} o\, f_i$ for all $i \geq 0$. We sometimes write $\mu : \Delta \to X$ as a reminder of the arrangement of μ's components Similarly, a colimit $\mu : \Delta \to X$ is a cocone with the property that if $\nu : \Delta \to X'$ is also a cocone then there exists a unique mediating arrow $k : X \to X'$ such that for all $i \geq 0,$, $\nu_i = k\, o\, \mu_i$. Colimits of $\omega - chains$ are sometimes referred to as $\omega - co\lim its$. Dually, an $\omega^{op} - chain$ in ***K*** is a diagram of the following form:

$$\Delta = D_o \xleftarrow{f_o} D_1 \xleftarrow{f_1} D_2 \xleftarrow{f_2} \ldots..$$

A cone $\mu : X \to \Delta$ of an $\omega^{op} - chain$ Δ is a ***K***-object X and a collection of **K**-arrows $\{\mu_i : D_i \mid i \geq 0\}$ such that for all $i \geq 0,$ $\mu_i = f_i\, o\, \mu_{i+1}$. An ω^{op} -limit of an $\omega^{op} - chain$ Δ is a cone $\mu : X \to \Delta$ with the property that if $\nu : X' \to \Delta$ is also a cone, then there exists a unique mediating arrow $k : X' \to X$ such that for all $i \geq 0, \mu_i\, o\, k = \nu_i$. We write $\perp_k$ (or just $\perp$) for the distinguish initial object of ***K***, when it has one, and $\perp \to A$ for the unique arrow from $\perp$ to each ***K***-object A. It is also convenient to write $\Delta^- = D_1 \xrightarrow{f_1} D_2 \xrightarrow{f_2} \ldots..$ to denote all of Δ except D_o and f_0. By analogy, μ^- is $\{\mu_i \mid i \geq 1\}$. For the images of Δ and μ under ***F*** we write

$$F(\Delta) = F(D_o) \xrightarrow{F(f_o)} F(D_1) \xrightarrow{F(f_1)} F(D_2) \xrightarrow{F(f_2)} \ldots..$$

and $F(\mu) = \{F(\mu_i) \mid i \geq 0\}$

We write F^i for the ***i***-fold iterated composition of ***F*** – that is, $F^o(f) = f, F^1(f) = F(f), F^2(f) = F(F(f))$,etc. With these definitions we can state that every monitonic function on a complete lattice has a least fixed point:

Lemma 1.4. Let ***K*** be a category with initial object $\perp$ and let $F: K \to K$ be a functor. Define the $\omega - chain\, \Delta$ by

$$\Delta = \perp \xrightarrow{!\perp \to F(\perp)} F(\perp) \xrightarrow{F(!\perp \to F(\perp))} F^2(\perp) \xrightarrow{F^2(!\perp \to F(\perp))} \ldots\ldots\ldots$$

If both $\mu : \Delta \to D$ and $F(\mu) : F(\Delta) \to F(D)$ are colimits, then (D,d) is an intial F-algebra, where $d : F(D) \to D$ is the mediating arrow from $F(\mu)$ to the cocone μ^{-}

Theorem 1.4 Let a DAG G given in which each node is a random variable, and let a discrete conditional probability distribution of each node given values of its parents in G be specified. Then the product of these conditional distributions yields a joint probability distribution P of the variables, and (G,P) satisfies the Markov condition.

Proof. Order the nodes according to an ancestral ordering. Let $X_1, X_2, \ldots\ldots.. X_n$ be the resultant ordering. Next define.

$$P(x_1, x_2, \ldots. x_n) = P(x_n \mid pa_n) P(x_{n-1} \mid Pa_{n-1}) \ldots$$
$$..P(x_2 \mid pa_2) P(x_1 \mid pa_1),$$

Where PA_i is the set of parents of X_i of in G and $P(x_i \mid pa_i)$ is the specified conditional probability distribution. First we show this does indeed yield a joint probability distribution. Clearly, $0 \le P(x_1, x_2, \ldots x_n) \le 1$ for all values of the variables. Therefore, to show we have a joint distribution, as the variables range through all their possible values, is equal to one. To that end, Specified conditional distributions are the conditional distributions they notationally represent in the joint distribution. Finally, we show the Markov condition is satisfied. To do this, we need show for $1 \le k \le n$ that whenever

$$P(pa_k) \neq 0, if \ \ P(nd_k \mid pa_k) \neq 0$$
$$and \ \ P(x_k \mid pa_k) \neq 0$$
$$then \ P(x_k \mid nd_k, pa_k) = P(x_k \mid pa_k),$$

Where ND_k is the set of nondescendents of X_k of in G. Since $PA_k \subseteq ND_k$, we need only show $P(x_k \mid nd_k) = P(x_k \mid pa_k)$. First for a given k, order the nodes so that all and only nondescendents of X_k precede X_k in the ordering. Note that this ordering depends on k, whereas the ordering in the first part of the proof does not. Clearly then

$$ND_k = \{X_1, X_2, \ldots. X_{k-1}\}$$
$$Let$$
$$D_k = \{X_{k+1}, X_{k+2}, \ldots. X_n\}$$

follows $\sum_{d_k}$

We define the m^{th} *cyclotomic field to be the field* $Q[x]/(\Phi_m(x))$ Where $\Phi_m(x)$ is the m^{th} cyclotomic polynomial. $Q[x]/(\Phi_m(x))$ $\Phi_m(x)$ *has degree* $\varphi(m)$ *over* Q *since* $\Phi_m(x)$ has degree $\varphi(m)$. *The roots of* $\Phi_m(x)$ are just the primitive m^{th} roots of unity, so the complex embeddings of $Q[x]/(\Phi_m(x))$ *are simply the* $\varphi(m)$ *maps*

$$\sigma_k : Q[x]/(\Phi_m(x)) \mapsto C,$$
$$1 \le k \prec m, (k,m) = 1, \quad where$$
$$\sigma_k(x) = \xi_m^k,$$

ξ_m being our fixed choice of primitive m^{th} root of unity. Note that $\xi_m^k \in Q(\xi_m)$ for every k; it follows that $Q(\xi_m) = Q(\xi_m^k)$ for all k relatively prime to m. In particular, the images of the σ_i coincide, so $Q[x]/(\Phi_m(x))$ *is Galois over* Q. *This means that we can write* $Q(\xi_m)$ *for* $Q[x]/(\Phi_m(x))$ *without much fear of ambiguity; we will do so from now on, the identification being* $\xi_m \mapsto x$. *One advantage of this is that one can easily talk about cyclotomic fields being extensions of one another,or intersections or compositums; all of these things take place considering them as subfield of* C. We now investigate some basic properties of cyclotomic fields. The first issue is whether or not they are all distinct; to determine this, we need to know which roots of unity lie in $Q(\xi_m)$ .Note, for example, that if m is odd, then $-\xi_m$ is a $2m^{th}$ root of unity. We will show that this is the only way in which one can obtain any non-m^{th} roots of unity.

LEMMA 1.5 If m divides n, then $Q(\xi_m)$ *is contained in* $Q(\xi_n)$

PROOF. Since $\xi^{n/m} = \xi_m$, *we have* $\xi_m \in Q(\xi_n)$, *so the result is clear*

LEMMA 1.6 If m and n are relatively prime, then

$$Q(\xi_m, \xi_n) = Q(\xi_{nm})$$

and

$$Q(\xi_m) \cap Q(\xi_n) = Q$$

(Recall the $Q(\xi_m,\xi_n)$ is the compositum of $Q(\xi_m)$ *and* $Q(\xi_n)$)

PROOF. One checks easily that $\xi_m\xi_n$ is a primitive mn^{th} root of unity, so that

$$Q(\xi_{mn}) \subseteq Q(\xi_m,\xi_n)$$
$$[Q(\xi_m,\xi_n):Q] \le [Q(\xi_m):Q][Q(\xi_n:Q]$$
$$= \varphi(m)\varphi(n) = \varphi(mn);$$

Since $[Q(\xi_{mn}):Q] = \varphi(mn);$ this implies that $Q(\xi_m,\xi_n) = Q(\xi_{nm})$ We know that $Q(\xi_m,\xi_n)$ has degree $\varphi(mn)$ over Q, so we must have

$$[Q(\xi_m,\xi_n):Q(\xi_m)] = \varphi(n)$$

and

$$[Q(\xi_m,\xi_n):Q(\xi_m)] = \varphi(m)$$

$$[Q(\xi_m):Q(\xi_m)\cap Q(\xi_n)] \ge \varphi(m)$$

And thus that $Q(\xi_m)\cap Q(\xi_n) = Q$

PROPOSITION 1.2 For any m and n

$$Q(\xi_m,\xi_n) = Q(\xi_{[m,n]})$$

And

$$Q(\xi_m)\cap Q(\xi_n) = Q(\xi_{(m,n)});$$

here $[m,n]$ and (m,n) denote the least common multiple and the greatest common divisor of m and n, respectively.

PROOF. Write $m = p_1^{e_1}......p_k^{e_k}$ *and* $p_1^{f_1}....p_k^{f_k}$ where the p_i are distinct primes. (We allow e_i *or* f_i to be zero)

$$Q(\xi_m) = Q(\xi_{p_1^{e_1}})Q(\xi_{p_2^{e_2}})...Q(\xi_{p_k^{e_k}})$$

and

$$Q(\xi_n) = Q(\xi_{p_1^{f_1}})Q(\xi_{p_2^{f_2}})...Q(\xi_{p_k^{f_k}})$$

Thus

$$Q(\xi_m,\xi_n) = Q(\xi_{p_1^{e_1}})........Q(\xi_{p_2^{e_k}})Q(\xi_{p_1^{f_1}})...Q(\xi_{p_k^{f_k}})$$
$$= Q(\xi_{p_1^{e_1}})Q(\xi_{p_1^{f_1}})...Q(\xi_{p_k^{e_k}})Q(\xi_{p_k^{f_k}})$$
$$= Q(\xi_{p_1^{\max(e_1,f_1)}}).......Q(\xi_{p_1^{\max(e_k,f_k)}})$$
$$= Q(\xi_{p_1^{\max(e_1,f_1)}........p_1^{\max(e_k,f_k)}})$$
$$= Q(\xi_{[m,n]});$$

An entirely similar computation shows that $Q(\xi_m)\cap Q(\xi_n) = Q(\xi_{(m,n)})$

Mutual information measures the information transferred when x_i is sent and y_i is received, and is defined as

$$I(x_i,y_i) = \log_2 \frac{P(x_i/y_i)}{P(x_i)} \; bits \qquad (1)$$

In a noise-free channel, **each** y_i is uniquely connected to the corresponding x_i , and so they constitute an input –output pair (x_i,y_i) for which

$$P(x_i/y_j) = 1 \; and \; I(x_i,y_j) = \log_2 \frac{1}{P(x_i)} \text{ bits;}$$

that is, the transferred information is equal to the self-information that corresponds to the input x_i In a very noisy channel, the output y_i and input x_i would be completely uncorrelated, and so $P(x_i/y_j) = P(x_i)$ and also $I(x_i,y_j) = 0;$ that is, there is no transference of information. In general, a given channel will operate between these two extremes. The mutual information is defined between the input and the output of a given channel. An average of the calculation of the mutual information for all input-output pairs of a given channel is the average mutual information:

$$I(X,Y) = \sum_{i.j} P(x_i,y_j)I(x_i,y_j) = \sum_{i.j} P(x_i,y_j)\log_2\left[\frac{P(x_i/y_j)}{P(x_i)}\right]$$

bits per symbol . This calculation is done over the input and output alphabets. The average mutual information. The following expressions are useful for modifying the mutual information expression:

$$P(x_i,y_j) = P(x_i/y_j)P(y_j) = P(y_j/x_i)P(x_i)$$

$$P(y_j) = \sum_i P(y_j/x_i)P(x_i)$$

$$P(x_i) = \sum_i P(x_i/y_j)P(y_j)$$

Then

$$I(X,Y) = \sum_{i.j} P(x_i, y_j)$$

$$= \sum_{i.j} P(x_i, y_j) \log_2 \left[\frac{1}{P(x_i)} \right]$$

$$-\sum_{i.j} P(x_i, y_j) \log_2 \left[\frac{1}{P(x_i/y_j)} \right]$$

$$\sum_{i.j} P(x_i, y_j) \log_2 \left[\frac{1}{P(x_i)} \right]$$

$$= \sum_i \left[P(x_i/y_j) P(y_j) \right] \log_2 \frac{1}{P(x_i)}$$

$$\sum_i P(x_i) \log_2 \frac{1}{P(x_i)} = H(X)$$

$$I(X,Y) = H(X) - H(X/Y)$$

Where $H(X/Y) = \sum_{i,j} P(x_i, y_j) \log_2 \frac{1}{P(x_i/y_j)}$

is usually called the equivocation. In a sense, the equivocation can be seen as the information lost in the noisy channel, and is a function of the backward conditional probability. The observation of an output symbol y_j provides $H(X) - H(X/Y)$ bits of information. This difference is the mutual information of the channel. *Mutual Information: Properties* Since

$$P(x_i/y_j) P(y_j) = P(y_j/x_i) P(x_i)$$

The mutual information fits the condition

$$I(X,Y) = I(Y,X)$$

And by interchanging input and output it is also true that

$$I(X,Y) = H(Y) - H(Y/X)$$

Where

$$H(Y) = \sum_j P(y_j) \log_2 \frac{1}{P(y_j)}$$

This last entropy is usually called the noise entropy. Thus, the information transferred through the channel is the difference between the output entropy and the noise entropy. Alternatively, it can be said that the channel mutual information is the difference between the number of bits needed for determining a given input symbol before knowing the corresponding output symbol, and the number of bits needed for determining a given input symbol after knowing the corresponding output symbol

$$I(X,Y) = H(X) - H(X/Y)$$

As the channel mutual information expression is a difference between two quantities, it seems that this parameter can adopt negative values. However, and is spite of the fact that for some $y_j, H(X/y_j)$ can be larger than $H(X)$, this is not possible for the average value calculated over all the outputs:

$$\sum_{i,j} P(x_i, y_j) \log_2 \frac{P(x_i/y_j)}{P(x_i)} = \sum_{i,j} P(x_i, y_j) \log_2 \frac{P(x_i, y_j)}{P(x_i)P(y_j)}$$

Then

$$-I(X,Y) = \sum_{i,j} P(x_i, y_j) \frac{P(x_i)P(y_j)}{P(x_i, y_j)} \leq 0$$

Because this expression is of the form

$$\sum_{i=1}^{M} P_i \log_2 \left(\frac{Q_i}{P_i}\right) \leq 0$$

The above expression can be applied due to the factor $P(x_i)P(y_j)$, which is the product of two probabilities, so that it behaves as the quantity Q_i, which in this expression is a dummy variable that fits the condition $\sum_i Q_i \leq 1$. It can be concluded that the average mutual information is a non-negative number. It can also be equal to zero, when the input and the output are independent of each other. A related entropy called the joint entropy is defined as

$$H(X,Y) = \sum_{i,j} P(x_i, y_j) \log_2 \frac{1}{P(x_i, y_j)}$$

$$= \sum_{i,j} P(x_i, y_j) \log_2 \frac{P(x_i)P(y_j)}{P(x_i, y_j)}$$

$$+ \sum_{i,j} P(x_i, y_j) \log_2 \frac{1}{P(x_i)P(y_j)}$$

Theorem 1.5: Entropies of the binary erasure channel (BEC) The BEC is defined with an alphabet of two inputs and three outputs, with symbol probabilities.

$P(x_1) = \alpha$ *and* $P(x_2) = 1 - \alpha$, and transition probabilities

$$P(y_3/x_2) = 1 - p \;\; and \;\; P(y_2/x_1) = 0,$$

$$and \;\; P(y_3/x_1) = 0$$

$$and \;\; P(y_1/x_2) = p$$

$$and \;\; P(y_3/x_2) = 1 - p$$

Lemma 1.7. Given an arbitrary restricted time-discrete, amplitude-continuous channel whose restrictions are determined by sets F_n and whose density functions exhibit no dependence on the state s, let n be a fixed positive integer, and $p(x)$ an arbitrary probability density function on Euclidean *n*-space. $p(y|x)$ for the density $p_n(y_1,...,y_n|x_1,...x_n)$ and F for F_n. For any real number a, let

$$A=\left\{(x,y):\log\frac{p(y|x)}{p(y)}>a\right\} \qquad (1)$$

Then for each positive integer u, there is a code (u,n,λ) such that

$$\lambda\le ue^{-a}+P\{(X,Y)\notin A\}+P\{X\notin F\} \qquad (2)$$

Where

$$P\{(X,Y)\in A\}=\int_A...\int p(x,y)dxdy, \qquad p(x,y)=p(x)p(y|x)$$

and

$$P\{X\in F\}=\int_F...\int p(x)dx$$

Proof: A sequence $x^{(1)}\in F$ *such that*

$$P\{Y\in A_{x^1}\,|\,X=x^{(1)}\}\ge 1-\varepsilon$$

$$where\ A_x=\{y:(x,y)\varepsilon A\};$$

Choose the decoding set B_1 to be $A_{x^{(1)}}$. Having chosen $x^{(1)},........,x^{(k-1)}$ and $B_1,...,B_{k-1}$, select $x^k\in F$ such that

$$P\left\{Y\in A_{x^{(k)}}-\bigcup_{i=1}^{k-1}B_i\,|\,X=x^{(k)}\right\}\ge 1-\varepsilon;$$

Set $B_k=A_{x^{(k)}}-\bigcup_{i=1}^{k-1}B_i$, If the process does not terminate in a finite number of steps, then the sequences $x^{(i)}$ and decoding sets B_i, $i=1,2,...,u$, form the desired code. Thus assume that the process terminates after t steps. (Conceivably $t=0$). We will show $t\ge u$ by showing that $\varepsilon\le te^{-a}+P\{(X,Y)\notin A\}+P\{X\notin F\}$. We proceed as follows.

Let

$$B=\bigcup_{j=1}^{t}B_j. \quad (If\ \ t=0,\ take\ \ B=\phi).\ \ Then$$

$$P\{(X,Y)\in A\}=\int_{(x,y)\in A}p(x,y)dx\,dy$$

$$=\int_x p(x)\int_{y\in A_x}p(y|x)dy\,dx$$

$$=\int_x p(x)\int_{y\in B\cap A_x}p(y|x)dy\,dx+\int_x p(x)$$

G. Algorithms

Ideals. Let A be a ring. Recall that an *ideal a* in A is a subset such that a is subgroup of A regarded as a group under addition;

$$a\in a, r\in A\Rightarrow ra\in A$$

The ideal generated by a subset S of A is the intersection of all ideals A containing a ----- it is easy to verify that this is in fact an ideal, and that it consist of all finite sums of the form $\sum r_is_i$ with $r_i\in A, s_i\in S$. When $S=\{s_1,.....,s_m\}$, we shall write $(s_1,.....,s_m)$ for the ideal it generates.

Let a and b be ideals in A. The set $\{a+b\,|\,a\in a, b\in b\}$ is an ideal, denoted by $a+b$. The ideal generated by $\{ab\,|\,a\in a, b\in b\}$ is denoted by ab. Note that $ab\subset a\cap b$. Clearly ab consists of all finite sums $\sum a_ib_i$ with $a_i\in a$ and $b_i\in b$, and if $a=(a_1,...,a_m)$ and $b=(b_1,...,b_n)$, then $ab=(a_1b_1,...,a_ib_j,...,a_mb_n)$.Let a be an ideal of A. The set of cosets of a in A forms a ring A/a, and $a\mapsto a+a$ is a homomorphism $\phi:A\mapsto A/a$. The map $b\mapsto\phi^{-1}(b)$ is a one to one correspondence between the ideals of A/a and the ideals of A containing a An ideal p if *prime* if $p\ne A$ and $ab\in p\Rightarrow a\in p$ or $b\in p$. Thus p is prime if and only if A/p is nonzero and has the property that $ab=0,\ \ b\ne 0\Rightarrow a=0,$ i.e., A/p is an integral domain. An ideal m is *maximal* if $m\ne| A$ and there does not exist an ideal n contained strictly between m and A. Thus m is maximal if and only if A/m has no proper nonzero ideals, and so is a field. Note that m maximal $\Rightarrow$ m prime. The ideals of $A\times B$ are all of the form $a\times b$, with a and b ideals in A and B. To see this, note that if c is an ideal in $A\times B$ and $(a,b)\in c$, then $(a,0)=(a,b)(1,0)\in c$ and $(0,b)=(a,b)(0,1)\in c$. This shows that $c=a\times b$ with

$$a=\{a\,|\,(a,b)\in c\ \ some\ \ b\in b\}$$

and

$$b=\{b\,|\,(a,b)\in c\ \ some\ \ a\in a\}$$

Let A be a ring. An A-algebra is a ring B together with a homomorphism $i_B:A\to B$. A *homomorphism of* A -algebra $B \to C$ is a homomorphism of rings $\varphi : B \to C$ such that $\varphi(i_B(a)) = i_C(a)$ for all $a \in A$. An A-algebra B is said to be *finitely generated* (or of *finite-type* over A) if there exist elements $x_1,...,x_n \in B$ such that every element of B can be expressed as a polynomial in the x_i with coefficients in $i(A)$, i.e., such that the homomorphism $A[X_1,...,X_n] \to B$ sending X_i to x_i is surjective. A ring homomorphism $A \to B$ is *finite*, and B is finitely generated as an A-module. Let k be a field, and let A be a k -algebra. If $1 \neq 0$ in A, then the map $k \to A$ is injective, we can identify k with its image, i.e., we can regard k as a subring of A . If 1=0 in a ring R, the R is the zero ring, i.e., $R = \{0\}$.

Polynomial rings. Let k be a field. A *monomial* in $X_1,...,X_n$ is an expression of the form $X_1^{a_1}...X_n^{a_n}$, $a_j \in N$. The *total degree* of the monomial is $\sum a_i$. We sometimes abbreviate it by X^α, $\alpha = (a_1,...,a_n) \in \square^n$. The elements of the polynomial ring $k[X_1,...,X_n]$ are finite sums $\sum c_{a_1....a_n} X_1^{a_1}...X_n^{a_n}$, $c_{a_1....a_n} \in k$, $a_j \in \square$

With the obvious notions of equality, addition and multiplication. Thus the monomials from basis for $k[X_1,...,X_n]$ as a k -vector space. The ring $k[X_1,...,X_n]$ is an integral domain, and the only units in it are the nonzero constant polynomials. A polynomial $f(X_1,...,X_n)$ is *irreducible* if it is nonconstant and has only the obvious factorizations, i.e., $f = gh \Rightarrow g$ or h is constant. **Division in** $k[X]$. The division algorithm allows us to divide a nonzero polynomial into another: let f and g be polynomials in $k[X]$ with $g \neq 0$; then there exist unique polynomials $q, r \in k[X]$ such that $f = qg + r$ with either $r = 0$ or deg r < deg g . Moreover, there is an algorithm for deciding whether $f \in (g)$, namely, find r and check whether it is zero. Moreover, the Euclidean algorithm allows to pass from finite set of generators for an ideal in $k[X]$ to a single generator by successively replacing each pair of generators with their greatest common divisor.

(Pure) ***lexicographic*** *ordering (lex).* Here monomials are ordered by lexicographic(dictionary) order. More precisely, let $\alpha = (a_1,...a_n)$ and $\beta = (b_1,...b_n)$ be two elements of $\square^n$; then $\alpha > \beta$ *and* $X^\alpha > X^\beta$ (lexicographic ordering) if, in the vector difference $\alpha - \beta \in \square$, the left most nonzero entry is positive. For example,

$XY^2 > Y^3Z^4$; $X^3Y^2Z^4 > X^3Y^2Z$. Note that this isn't quite how the dictionary would order them: it would put $XXXYYZZZZ$ after $XXXYYZ$. *Graded reverse lexicographic order (grevlex).* Here monomials are ordered by total degree, with ties broken by reverse lexicographic ordering. Thus, $\alpha > \beta$ if $\sum a_i > \sum b_i$, or $\sum a_i = \sum b_i$ and in $\alpha - \beta$ the right most nonzero entry is negative. For example:

$X^4Y^4Z^7 > X^5Y^5Z^4$ *(total degree greater)*

$XY^5Z^2 > X^4YZ^3$, $X^5YZ > X^4YZ^2$.

Orderings on $k[X_1,...X_n]$ **.** Fix an ordering on the monomials in $k[X_1,...X_n]$. Then we can write an element f of $k[X_1,...X_n]$ in a canonical fashion, by re-ordering its elements in decreasing order. For example, we would write

$$f = 4XY^2Z + 4Z^2 - 5X^3 + 7X^2Z^2$$

as

$$f = -5X^3 + 7X^2Z^2 + 4XY^2Z + 4Z^2 \quad (lex)$$

or

$$f = 4XY^2Z + 7X^2Z^2 - 5X^3 + 4Z^2 \quad (grevlex)$$

Let $\sum a_\alpha X^\alpha \in k[X_1,...,X_n]$, in decreasing order:

$$f = a_{\alpha_0} X^{\alpha_0} +_{\alpha_1} X^{\alpha_1} + ..., \qquad \alpha_0 > \alpha_1 > ..., \quad \alpha_0 \neq 0$$

Then we define.

- The *multidegree* of f to be multdeg(f)= α_0 ;
- The *leading coefficient of* f to be *LC(* f *)=* a_{α_0} ;
- The *leading monomial of* f to be *LM(* f *)* = X^{α_0} ;
- The *leading term of* f to be *LT(* f *)* = $a_{\alpha_0} X^{\alpha_0}$

For the polynomial $f = 4XY^2Z + ...,$ the multidegree is (1,2,1), the leading coefficient is 4, the leading monomial is XY^2Z , and the leading term is $4XY^2Z$. **The division algorithm in**

$k[X_1,\ldots X_n]$. Fix a monomial ordering in $\square^2$. Suppose given a polynomial f and an ordered set $(g_1,\ldots g_s)$ of polynomials; the division algorithm then constructs polynomials $a_1,\ldots a_s$ and r such that $f = a_1g_1+\ldots+a_sg_s+r$ Where either $r=0$ or no monomial in r is divisible by any of $LT(g_1),\ldots,LT(g_s)$ **Step 1:** If $LT(g_1)\,|\,LT(f)$, divide g_1 into f to get

$$f = a_1g_1+h, \qquad a_1=\frac{LT(f)}{LT(g_1)}\in k[X_1,\ldots,X_n]$$

If $LT(g_1)\,|\,LT(h)$, repeat the process until $f = a_1g_1+f_1$ (different a_1) with $LT(f_1)$ not divisible by $LT(g_1)$. Now divide g_2 into f_1, and so on, until $f = a_1g_1+\ldots+a_sg_s+r_1$ With $LT(r_1)$ not divisible by any $LT(g_1),\ldots LT(g_s)$ **Step 2:** Rewrite $r_1 = LT(r_1)+r_2$, and repeat Step 1 with r_2 for f: $f = a_1g_1+\ldots+a_sg_s+LT(r_1)+r_3$ (different a_i's) **Monomial ideals.** In general, an ideal a will contain a polynomial without containing the individual terms of the polynomial; for example, the ideal $a=(Y^2-X^3)$ contains Y^2-X^3 but not Y^2 or X^3.

DEFINITION 1.5. An ideal a is *monomial* if $\sum c_\alpha X^\alpha \in a \Rightarrow X^\alpha \in a$ all α with $c_\alpha \neq 0$.

PROPOSITION 1.3. Let a be a *monomial ideal*, and let $A=\{\alpha \mid X^\alpha \in a\}$. Then A satisfies the condition $\alpha \in A,\ \beta \in \square^n \Rightarrow \alpha+\beta \in$ $(*)$ And a is the k-subspace of $k[X_1,\ldots,X_n]$ generated by the $X^\alpha, \alpha\in A$. Conversely, of A is a subset of $\square^n$ satisfying $(*)$, then the k-subspace a of $k[X_1,\ldots,X_n]$ generated by $\{X^\alpha \mid \alpha\in A\}$ is a monomial ideal.

PROOF. It is clear from its definition that a monomial ideal a is the k-subspace of $k[X_1,\ldots,X_n]$ generated by the set of monomials it contains. If $X^\alpha \in a$ and $X^\beta \in k[X_1,\ldots,X_n]$.

If a permutation is chosen uniformly and at random from the $n!$ possible permutations in S_n, then the counts $C_j^{(n)}$ of cycles of length j are dependent random variables. The joint distribution of $C^{(n)}=(C_1^{(n)},\ldots,C_n^{(n)})$ follows from Cauchy's formula, and is given by

$$P[C^{(n)}=c]=\frac{1}{n!}N(n,c)=1\left\{\sum_{j=1}^{n} jc_j=n\right\}\prod_{j=1}^{n}\left(\frac{1}{j}\right)^{c_j}\frac{1}{c_j!}, \tag{1.1}$$

for $c\in\square_+^n$.

Lemma1.7 For nonnegative integers $m_{1,\ldots,}m_n$,

$$E\left(\prod_{j=1}^{n}(C_j^{(n)})^{[m_j]}\right)=\left(\prod_{j=1}^{n}\left(\frac{1}{j}\right)^{m_j}\right)1\left\{\sum_{j=1}^{n} jm_j\le n\right\} \tag{1.4}$$

Proof. This can be established directly by exploiting cancellation of the form $c_j^{[m_j]}/c_j^{!}=1/(c_j-m_j)!$ when $c_j\ge m_j$, which occurs between the ingredients in Cauchy's formula and the falling factorials in the moments. Write $m=\sum jm_j$. Then, with the first sum indexed by $c=(c_1,\ldots c_n)\in\square_+^n$ and the last sum indexed by $d=(d_1,\ldots,d_n)\in\square_+^n$ via the correspondence $d_j=c_j-m_j$, we have

$$E\left(\prod_{j=1}^{n}(C_j^{(n)})^{[m_j]}\right)=\sum_c P[C^{(n)}=c]\prod_{j=1}^{n}(c_j)^{[m_j]}$$

$$=\sum_{c:c_j\ge m_j \text{ for all } j} 1\left\{\sum_{j=1}^{n} jc_j=n\right\}\prod_{j=1}^{n}\frac{(c_j)^{[m_j]}}{j^{c_j}c_j!}$$

$$=\prod_{j=1}^{n}\frac{1}{j^{m_j}}\sum_d 1\left\{\sum_{j=1}^{n} jd_j=n-m\right\}\prod_{j=1}^{n}\frac{1}{j^{d_j}(d_j)!}$$

This last sum simplifies to the indicator $1(m\le n)$, corresponding to the fact that if $n-m\ge 0$, then $d_j=0$ for $j>n-m$, and a random permutation in S_{n-m} must have some cycle structure $(d_1,\ldots,d_{n-m})$. The moments of $C_j^{(n)}$ follow immediately as

$$E(C_j^{(n)})^{[r]}=j^{-r}1\{jr\le n\} \tag{1.2}$$

We note for future reference that (1.4) can also be written in the form

$$E\left(\prod_{j=1}^{n}(C_j^{(n)})^{[m_j]}\right)=E\left(\prod_{j=1}^{n}Z_j^{[m_j]}\right)1\left\{\sum_{j=1}^{n} jm_j\le n\right\}, \tag{1.3}$$

Where the Z_j are independent Poisson-distribution random variables that satisfy $E(Z_j) = 1/j$

The marginal distribution of cycle counts provides a formula for the joint distribution of the cycle counts C_j^n, we find the distribution of C_j^n using a combinatorial approach combined with the inclusion-exclusion formula.

Lemma 1.8. For $1 \le j \le n$,

$$P[C_j^{(n)} = k] = \frac{j^{-k}}{k!} \sum_{l=0}^{[n/j]-k} (-1)^l \frac{j^{-l}}{l!} \qquad (1.1)$$

Proof. Consider the set I of all possible cycles of length j, formed with elements chosen from $\{1,2,\ldots n\}$, so that $|I| = n^{[j]/j}$. For each $\alpha \in I$, consider the "property" G_α of having α; that is, G_α is the set of permutations $\pi \in S_n$ such that α is one of the cycles of π. We then have $|G_\alpha| = (n-j)!$, since the elements of $\{1,2,\ldots,n\}$ not in α must be permuted among themselves. To use the inclusion-exclusion formula we need to calculate the term S_r, which is the sum of the probabilities of the r-fold intersection of properties, summing over all sets of r distinct properties. There are two cases to consider. If the r properties are indexed by r cycles having no elements in common, then the intersection specifies how rj elements are moved by the permutation, and there are $(n-rj)!1(rj \le n)$ permutations in the intersection. There are $n^{[rj]}/(j^r r!)$ such intersections. For the other case, some two distinct properties name some element in common, so no permutation can have both these properties, and the r-fold intersection is empty. Thus

$$S_r = (n-rj)!1(rj \le n) \times \frac{n^{[rj]}}{j^r r!} \frac{1}{n!} = 1(rj \le n) \frac{1}{j^r r!}$$

Finally, the inclusion-exclusion series for the number of permutations having exactly k properties is

$$\sum_{l \ge 0} (-1)^l \binom{k+l}{l} S_{k+l},$$

Which simplifies to (1.1) Returning to the original hat-check problem, we substitute j=1 in (1.1) to obtain the distribution of the number of fixed points of a random permutation. For $k = 0,1,\ldots,n,$

$$P[C_1^{(n)} = k] = \frac{1}{k!} \sum_{l=0}^{n-k} (-1)^l \frac{1}{l!}, \qquad (1.2)$$

and the moments of $C_1^{(n)}$ follow from (1.2) with $j = 1$. In particular, for $n \ge 2$, the mean and variance of $C_1^{(n)}$ are both equal to 1. The joint distribution of $(C_1^{(n)},\ldots,C_b^{(n)})$ for any $1 \le b \le n$ has an expression similar to (1.7); this too can be derived by inclusion-exclusion. For any $c = (c_1,\ldots,c_b) \in \mathbb{N}_+^b$ with $m = \sum ic_i$,

$$P[(C_1^{(n)},\ldots,C_b^{(n)}) = c] = \left\{ \prod_{i=1}^{b} \left(\frac{1}{i}\right)^{c_i} \frac{1}{c_i!} \right\} \sum_{\substack{l \ge 0 \text{ with} \\ \sum il_i \le n-m}} (-1)^{l_1+\ldots+l_b} \prod_{i=1}^{b} \left(\frac{1}{i}\right)^{l_i} \frac{1}{l_i!} \qquad (1.3)$$

The joint moments of the first b counts $C_1^{(n)},\ldots,C_b^{(n)}$ can be obtained directly from (1.2) and (1.3) by setting $m_{b+1} = \ldots = m_n = 0$

The limit distribution of cycle counts

It follows immediately from Lemma 1.2 that for each fixed j, as $n \to \infty$,

$$P[C_j^{(n)} = k] \to \frac{j^{-k}}{k!} e^{-1/j}, \quad k = 0,1,2,\ldots,$$

So that $C_j^{(n)}$ converges in distribution to a random variable Z_j having a Poisson distribution with mean $1/j$; we use the notation $C_j^{(n)} \to_d Z_j$ where $Z_j \sim P_o(1/j)$ to describe this. Infact, the limit random variables are independent.

Theorem 1.6 The process of cycle counts converges in distribution to a Poisson process of $\mathbb{N}$ with intensity j^{-1}. That is, as $n \to \infty$,

$$(C_1^{(n)}, C_2^{(n)},\ldots) \to_d (Z_1, Z_2,\ldots) \qquad (1.1)$$

Where the $Z_j, j = 1,2,\ldots,$ are independent Poisson-distributed random variables with $E(Z_j) = \frac{1}{j}$

Proof. To establish the converges in distribution one shows that for each fixed $b \ge 1$, as $n \to \infty$,

$$P[(C_1^{(n)},\ldots,C_b^{(n)}) = c] \to P[(Z_1,\ldots,Z_b) = c]$$

Error rates

The proof of Theorem says nothing about the rate of convergence. Elementary analysis can be used to estimate this rate when $b = 1$. Using properties of alternating series with decreasing terms, for $k = 0,1,\ldots,n,$

$$\frac{1}{k!}\left(\frac{1}{(n-k+1)!} - \frac{1}{(n-k+2)!}\right) \le \left|P[C_1^{(n)} = k] - P[Z_1 = k]\right|$$
$$\le \frac{1}{k!(n-k+1)!}$$

It follows that

$$\frac{2^{n+1}}{(n+1)!}\frac{n}{n+2} \le \sum_{k=0}^{n}\left|P[C_1^{(n)} = k] - P[Z_1 = k]\right| \le \frac{2^{n+1}-1}{(n+1)!} \quad (1.11)$$

Since

$$P[Z_1 > n] = \frac{e^{-1}}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \ldots\right) < \frac{1}{(n+1)!},$$

We see from (1.11) that the total variation distance between the distribution $L(C_1^{(n)})$ of $C_1^{(n)}$ and the distribution $L(Z_1)$ of Z_1

Establish the asymptotics of $\mathrm{P}\left[A_n(C^{(n)})\right]$ under conditions (A_0) and (B_{01}), where

$$A_n(C^{(n)}) = \bigcap_{1\le i\le n}\ \bigcap_{r_i'+1\le j\le r_i}\left\{C_{ij}^{(n)} = 0\right\},$$

and $\zeta_i = (r_i'/r_{id}) - 1 = O(i^{-g'})$ as $i \to \infty$, for some $g' > 0$. We start with the expression

$$P[A_n(C^{(n)})] = \frac{P[T_{0m}(Z') = n]}{P[T_{0m}(Z) = n]}$$

$$\prod_{\substack{1\le i\le n\\ r_i'+1\le j\le r_i}}\left\{1 - \frac{\theta}{ir_i}(1+E_{i0})\right\} \quad (1.1)$$

$$P[T_{0n}(Z') = n]$$
$$= \frac{\theta d}{n}\exp\left\{\sum_{i\ge1}[\log(1+i^{-1}\theta d) - i^{-1}\theta d]\right\}$$
$$\left\{1 + O(n^{-1}\varphi_{\{1,2,7\}}'(n))\right\} \quad (1.2)$$

and

$$P[T_{0n}(Z') = n]$$
$$= \frac{\theta d}{n}\exp\left\{\sum_{i\ge1}[\log(1+i^{-1}\theta d) - i^{-1}\theta d]\right\}$$
$$\left\{1 + O(n^{-1}\varphi_{\{1,2,7\}}(n))\right\} \quad (1.3)$$

Where $\varphi_{\{1,2,7\}}'(n)$ refers to the quantity derived from Z'. It thus follows that $P[A_n(C^{(n)})] \sim Kn^{-\theta(1-d)}$ for a constant K, depending on Z and the r_i' and computable explicitly from (1.1) – (1.3), if Conditions (A_0) and (B_{01}) are satisfied and if $\zeta_i^* = O(i^{-g'})$ from some $g' > 0$, since, under these circumstances, both $n^{-1}\varphi_{\{1,2,7\}}'(n)$ and $n^{-1}\varphi_{\{1,2,7\}}(n)$ tend to zero as $n \to \infty$. In particular, for polynomials and square free polynomials, the relative error in this asymptotic approximation is of order n^{-1} if $g' > 1$.

For $0 \le b \le n/8$ and $n \ge n_0$, with n_0

$$d_{TV}(L(C[1,b]), L(Z[1,b]))$$
$$\le d_{TV}(L(\tilde{C}[1,b]), L(\tilde{Z}[1,b]))$$
$$\le \varepsilon_{\{7,7\}}(n,b),$$

Where $\varepsilon_{\{7,7\}}(n,b) = O(b/n)$ under Conditions $(A_0),(D_1)$ and (B_{11}) Since, by the Conditioning Relation,

$$L(\tilde{C}[1,b] \mid T_{0b}(C) = l) = L(\tilde{Z}[1,b] \mid T_{0b}(Z) = l),$$

It follows by direct calculation that

$$d_{TV}(L(\tilde{C}[1,b]), L(\tilde{Z}[1,b]))$$
$$= d_{TV}(L(T_{0b}(C)), L(T_{0b}(Z)))$$
$$= \max_A \sum_{r\in A} P[T_{0b}(Z) = r]$$
$$\left\{1 - \frac{P[T_{bn}(Z) = n-r]}{P[T_{0n}(Z) = n]}\right\} \quad (1.4)$$

Suppressing the argument Z from now on, we thus obtain

$$d_{TV}(L(\tilde{C}[1,b]), L(\tilde{Z}[1,b]))$$
$$= \sum_{r\ge0} P[T_{0b} = r]\left\{1 - \frac{P[T_{bn} = n-r]}{P[T_{0n} = n]}\right\}_+$$
$$\le \sum_{r>n/2} P[T_{0b} = r] + \sum_{r=0}^{[n/2]} \frac{P[T_{0b} = r]}{P[T_{0b} = n]}$$
$$\times\left\{\sum_{s=0}^{n} P[T_{0b} = s](P[T_{bn} = n-s] - P[T_{bn} = n-r]\right\}_+$$

$$\le \sum_{r>n/2} P[T_{0b} = r] + \sum_{r=0}^{[n/2]} P[T_{0b} = r]$$

$$\times \sum_{s=0}^{[n/2]} P[T_{0b}=s] \frac{\{P[T_{bn}=n-s]-P[T_{bn}=n-r]\}}{P[T_{0n}=n]}$$

$$+\sum_{s=0}^{[n/2]} P[T_{0b}=r] \sum_{s=[n/2]+1}^{n} P[T=s]P[T_{bn}=n-s]/P[T_{0n}=n]$$

The first sum is at most $2n^{-1}ET_{0b}$; the third is bound by

$$(\max_{n/2<s\le n} P[T_{0b}=s])/P[T_{0n}=n]$$

$$\le \frac{2\varepsilon_{\{10.5(1)\}}(n/2,b)}{n}\frac{3n}{\theta P_\theta[0,1]},$$

$$\frac{3n}{\theta P_\theta[0,1]} 4n^{-2}\phi^*_{\{10.8\}}(n)\sum_{r=0}^{[n/2]} P[T_{0b}=r]\sum_{s=0}^{[n/2]} P[T_{0b}=s]\frac{1}{2}|r-s|$$

$$\le \frac{12\phi^*_{\{10.8\}}(n)}{\theta P_\theta[0,1]}\frac{ET_{0b}}{n}$$

Hence we may take

$$\varepsilon_{\{7,7\}}(n,b) = 2n^{-1}ET_{0b}(Z)\left\{1+\frac{6\phi^*_{\{10.8\}}(n)}{\theta P_\theta[0,1]}\right\}P$$

$$+\frac{6}{\theta P_\theta[0,1]}\varepsilon_{\{10.5(1)\}}(n/2,b) \qquad (1.5)$$

Required order under Conditions $(A_0),(D_1)$ and (B_{11}), if $S(\infty)<\infty$. If not, $\phi^*_{\{10.8\}}(n)$ can be replaced by $\phi^*_{\{10.11\}}(n)$ in the above, which has the required order, without the restriction on the r_i implied by $S(\infty)<\infty$. Examining the Conditions $(A_0),(D_1)$ and (B_{11}), it is perhaps surprising to find that (B_{11}) is required instead of just (B_{01}); that is, that we should need $\sum_{l\ge 2} l\varepsilon_{il} = O(i^{-a_1})$ to hold for some $a_1>1$. A first observation is that a similar problem arises with the rate of decay of ε_{i1} as well. For this reason, n_1 is replaced by n_1. This makes it possible to replace condition (A_1) by the weaker pair of conditions (A_0) and (D_1) in the eventual assumptions needed for $\varepsilon_{\{7,7\}}(n,b)$ to be of order $O(b/n)$; the decay rate requirement of order $i^{-1-\gamma}$ is shifted from ε_{i1} itself to its first difference. This is needed to obtain the right approximation error for the random mappings example. However, since all the classical applications make far more stringent assumptions about the $\varepsilon_{i1}, l\ge 2$, than are made in (B_{11}). The critical point of the proof is seen where the initial estimate of the difference $P[T^{(m)}_{bn}=s]-P[T^{(m)}_{bn}=s+1]$. The factor $\varepsilon_{\{10.10\}}(n)$, which should be small, contains a far tail element from n_1 of the form $\phi_1^\theta(n)+u_1^*(n)$, which is only small if $a_1>1$, being otherwise of order $O(n^{1-a_1+\delta})$ for any $\delta>0$, since $a_2>1$ is in any case assumed. For $s\ge n/2$, this gives rise to a contribution of order $O(n^{-1-a_1+\delta})$ in the estimate of the difference $P[T_{bn}=s]-P[T_{bn}=s+1]$, which, in the remainder of the proof, is translated into a contribution of order $O(tn^{-1-a_1+\delta})$ for differences of the form $P[T_{bn}=s]-P[T_{bn}=s+1]$, finally leading to a contribution of order $bn^{-a_1+\delta}$ for any $\delta>0$ in $\varepsilon_{\{7.7\}}(n,b)$. Some improvement would seem to be possible, defining the function g by $g(w)=1_{\{w=s\}}-1_{\{w=s+t\}}$, differences that are of the form $P[T_{bn}=s]-P[T_{bn}=s+t]$ can be directly estimated, at a cost of only a single contribution of the form $\phi_1^\theta(n)+u_1^*(n)$. Then, iterating the cycle, in which one estimate of a difference in point probabilities is improved to an estimate of smaller order, a bound of the form

$$|P[T_{bn}=s]-P[T_{bn}=s+t]| = O(n^{-2}t+n^{-1-a_1+\delta})$$

for any $\delta>0$ could perhaps be attained, leading to a final error estimate in order $O(bn^{-1}+n^{-a_1+\delta})$ for any $\delta>0$, to replace $\varepsilon_{\{7.7\}}(n,b)$. This would be of the ideal order $O(b/n)$ for large enough b, but would still be coarser for small b.

With b and n as in the previous section, we wish to show that

$$\left|d_{TV}(L(C[1,b]),L(Z[1,b]))-\frac{1}{2}(n+1)^{-1}|1-\theta|E|T_{0b}-ET_{0b}|\right|$$

$$\le \varepsilon_{\{7,8\}}(n,b),$$

Where $\varepsilon_{\{7.8\}}(n,b)=O(n^{-1}b[n^{-1}b+n^{-\beta_{12}+\delta}])$ for any $\delta>0$ under Conditions $(A_0),(D_1)$ and (B_{12}), with β_{12}. The proof uses sharper estimates. As before, we begin with the formula

$$d_{TV}(L(\tilde{C}[1,b]), L(\tilde{Z}[1,b]))$$
$$= \sum_{r\ge 0} P[T_{0b}=r]\left\{1-\frac{P[T_{bn}=n-r]}{P[T_{0n}=n]}\right\}_+$$

Now we observe that

$$\left|\sum_{r\ge 0} P[T_{0b}=r]\left\{1-\frac{P[T_{bn}=n-r]}{P[T_{0n}=n]}\right\}_+ - \sum_{r=0}^{[n/2]}\frac{P[T_{0b}=r]}{P[T_{0n}=n]}\right|$$
$$\times\left|\sum_{s=[n/2]+1}^{n} P[T_{0b}=s](P[T_{bn}=n-s]-P[T_{bn}=n-r])\right|$$
$$\le 4n^{-2}ET_{0b}^2 + (\max_{n/2<s\le n} P[T_{0b}=s])/P[T_{0n}=n]$$
$$+P[T_{0b}>n/2]$$
$$\le 8n^{-2}ET_{0b}^2 + \frac{3\varepsilon_{\{10.5(2)\}}(n/2,b)}{\theta P_\theta[0,1]}, \qquad (1.1)$$

We have

$$\left|\sum_{r=0}^{[n/2]}\frac{P[T_{0b}=r]}{P[T_{0n}=n]}\right.$$
$$\times\left(\left\{\sum_{s=0}^{[n/2]} P[T_{0b}=s](P[T_{bn}=n-s]-P[T_{bn}=n-r]\right\}_+\right.$$
$$\left.\left.-\left\{\sum_{s=0}^{[n/2]} P[T_{0b}=s]\frac{(s-r)(1-\theta)}{n+1}P[T_{0n}=n]\right\}_+\right)\right|$$

$$\le \frac{1}{n^2P[T_{0n}=n]}\sum_{r\ge 0}P[T_{0b}=r]\sum_{s\ge 0}P[T_{0b}=s]|s-r|$$
$$\times\left\{\varepsilon_{\{10.14\}}(n,b)+2(r\vee s)|1-\theta|n^{-1}\left\{K_0\theta+4\phi^*_{\{10.8\}}(n)\right\}\right\}$$
$$\le \frac{6}{\theta nP_\theta[0,1]}ET_{0b}\varepsilon_{\{10.14\}}(n,b)$$
$$+4|1-\theta|n^{-2}ET_{0b}^2\left\{K_0\theta+4\phi^*_{\{10.8\}}(n)\right\}$$
$$\left(\frac{3}{\theta nP_\theta[0,1]}\right)\Big\}, \qquad (1.2)$$

The approximation in (1.2) is further simplified by noting that

$$\sum_{r=0}^{[n/2]}P[T_{0b}=r]\left|\left\{\sum_{s=0}^{[n/2]}P[T_{0b}=s]\frac{(s-r)(1-\theta)}{n+1}\right\}_+\right.$$
$$\left.-\left\{\sum_{s=0}P[T_{0b}=s]\frac{(s-r)(1-\theta)}{n+1}\right\}_+\right|$$
$$\le\sum_{r=0}^{[n/2]}P[T_{0b}=r]\sum_{s>[n/2]}P[T_{0b}=s]\frac{(s-r)|1-\theta|}{n+1}$$
$$\le|1-\theta|n^{-1}E(T_{0b}1\{T_{0b}>n/2\})\le 2|1-\theta|n^{-2}ET_{0b}^2, \qquad (1.3)$$

and then by observing that

$$\sum_{r>[n/2]}P[T_{0b}=r]\left\{\sum_{s\ge 0}P[T_{0b}=s]\frac{(s-r)(1-\theta)}{n+1}\right\}$$
$$\le n^{-1}|1-\theta|(ET_{0b}P[T_{0b}>n/2]+E(T_{0b}1\{T_{0b}>n/2\}))$$
$$\le 4|1-\theta|n^{-2}ET_{0b}^2 \qquad (1.4)$$

Combining the contributions of (1.2) –(1.3), we thus tha find

$$\left|\ d_{TV}(L(\tilde{C}[1,b]),L(\tilde{Z}[1,b]))\right.$$
$$\left.-(n+1)^{-1}\sum_{r\ge 0}P[T_{0b}=r]\left\{\sum_{s\ge 0}P[T_{0b}=s](s-r)(1-\theta)\right\}_+\ \right|$$
$$\le\varepsilon_{\{7.8\}}(n,b)$$
$$=\frac{3}{\theta P_\theta[0,1]}\left\{\varepsilon_{\{10.5(2)\}}(n/2,b)+2n^{-1}ET_{0b}\varepsilon_{\{10.14\}}(n,b)\right\}$$
$$+2n^{-2}ET_{0b}^2\left\{4+3|1-\theta|+\frac{24|1-\theta|\phi^*_{\{10.8\}}(n)}{\theta P_\theta[0,1]}\right\} \qquad (1.5)$$

The quantity $\varepsilon_{\{7.8\}}(n,b)$ is seen to be of the order claimed under Conditions $(A_0),(D_1)$ and (B_{12}), provided that $S(\infty)<\infty$; this supplementary condition can be removed if $\phi^*_{\{10.8\}}(n)$ is replaced by $\phi^*_{\{10.11\}}(n)$ in the definition of $\varepsilon_{\{7.8\}}(n,b)$, has the required order without the restriction on the r_i implied by assuming that $S(\infty)<\infty$. Finally, a direct calculation now shows that

$$\sum_{r\ge 0}P[T_{0b}=r]\left\{\sum_{s\ge 0}P[T_{0b}=s](s-r)(1-\theta)\right\}_+$$
$$=\frac{1}{2}|1-\theta|E|T_{0b}-ET_{0b}|$$

Example 1.0. Consider the point $O=(0,\ldots,0)\in\mathbb{R}^n$. For an arbitrary vector r, the coordinates of the point $x=O+r$ are equal to the respective coordinates of the vector $r: x=(x^1,\ldots x^n)$ and $r=(x^1,\ldots,x^n)$. The vector r such as in the example is called the position vector or the radius vector of the point x. (Or, in greater detail: r is the radius-vector of x w.r.t an origin O). Points are frequently specified by their radius-vectors. This presupposes the choice of O as the "standard origin". Let us summarize. We have considered $\mathbb{R}^n$ and interpreted its elements in two ways: as points and as vectors. Hence we may say that we leading with the two copies of $\mathbb{R}^n$: $\mathbb{R}^n$ = {points}, $\mathbb{R}^n$ = {vectors}

Operations with vectors: multiplication by a number, addition. Operations with points and vectors: adding a vector to a point (giving a point), subtracting two points (giving a vector). $\mathbb{R}^n$ treated in this way is called an *n-dimensional affine space*. *(*An "abstract" affine space is a pair of sets , the set of points and the set of vectors so that the operations as above are defined axiomatically). Notice that vectors in an affine space are also known as "free vectors". Intuitively, they are not fixed at points and "float freely" in space. From $\mathbb{R}^n$ considered as an affine space we can precede in two opposite directions: $\mathbb{R}^n$ as an Euclidean space $\Leftarrow$ $\mathbb{R}^n$ as an affine space $\Rightarrow$ $\mathbb{R}^n$ as a manifold.Going to the left means introducing some extra structure which will make the geometry richer. Going to the right means forgetting about part of the affine structure; going further in this direction will lead us to the so-called "smooth (or differentiable) manifolds". The theory of differential forms does not require any extra geometry. So our natural direction is to the right. The Euclidean structure, however, is useful for examples and applications. So let us say a few words about it:

Remark 1.0. *Euclidean geometry.* In $\mathbb{R}^n$ considered as an affine space we can already do a good deal of geometry. For example, we can consider lines and planes, and quadric surfaces like an ellipsoid. However, we cannot discuss such things as "lengths", "angles" or "areas" and "volumes". To be able to do so, we have to introduce some more definitions, making $\mathbb{R}^n$ a Euclidean space. Namely, we define the length of a vector $a=(a^1,...,a^n)$ to be

$$|a|:=\sqrt{(a^1)^2+...+(a^n)^2} \qquad (1)$$

After that we can also define distances between points as follows:

$$d(A,B):=|\overline{AB}| \qquad (2)$$

One can check that the distance so defined possesses natural properties that we expect: is it always non-negative and equals zero only for coinciding points; the distance from A to B is the same as that from B to A (symmetry); also, for three points, A, B and C, we have $d(A,B)\le d(A,C)+d(C,B)$ (the "triangle inequality"). To define angles, we first introduce the scalar product of two vectors

$$(a,b):=a^1b^1+...+a^nb^n \qquad (3)$$

Thus $|a|=\sqrt{(a,a)}$. The scalar product is also denote by dot: $a.b=(a,b)$, and hence is often referred to as the "dot product" . Now, for nonzero vectors, we define the angle between them by the equality

$$\cos\alpha:=\frac{(a,b)}{|a||b|} \qquad (4)$$

The angle itself is defined up to an integral multiple of 2π . For this definition to be consistent we have to ensure that the r.h.s. of (4) does not exceed 1 by the absolute value. This follows from the inequality

$$(a,b)^2\le|a|^2|b|^2 \qquad (5)$$

known as the Cauchy–Bunyakovsky–Schwarz inequality (various combinations of these three names are applied in different books). One of the ways of proving (5) is to consider the scalar square of the linear combination $a+tb,$ where $t\in R$. As $(a+tb,a+tb)\ge 0$ is a quadratic polynomial in t which is never negative, its discriminant must be less or equal zero. Writing this explicitly yields (5). The triangle inequality for distances also follows from the inequality (5).

Example 1.1. Consider the function $f(x)=x^i$ (the i-th coordinate). The linear function dx^i (the differential of x^i) applied to an arbitrary vector h is simply h^i.From these examples follows that we can rewrite df as

$$df=\frac{\partial f}{\partial x^1}dx^1+...+\frac{\partial f}{\partial x^n}dx^n, \qquad (1)$$

which is the standard form. Once again: the partial derivatives in (1) are just the coefficients (depending on x); $dx^1,dx^2,...$ are linear functions giving on an arbitrary vector h its coordinates $h^1,h^2,...,$ respectively. Hence

$$df(x)(h)=\partial_{hf(x)}=\frac{\partial f}{\partial x^1}h^1+ ...+\frac{\partial f}{\partial x^n}h^n, \qquad (2)$$

Theorem 1.7. Suppose we have a parametrized curve $t\mapsto x(t)$ passing through $x_0\in\mathbb{R}^n$ at $t=t_0$ and with the velocity vector $x(t_0)=\upsilon$ Then

$$\frac{df(x(t))}{dt}(t_0)=\partial_\upsilon f(x_0)=df(x_0)(\upsilon) \qquad (1)$$

Proof. Indeed, consider a small increment of the parameter $t : t_0 \mapsto t_0 + \Delta t$, Where $\Delta t \mapsto 0$. On the other hand, we have $f(x_0 + h) - f(x_0) = df(x_0)(h) + \beta(h)|h|$ for an arbitrary vector h , where $\beta(h) \to 0$ when $h \to 0$. Combining it together, for the increment of $f(x(t))$ we obtain

$$f(x(t_0 + \Delta t) - f(x_0)$$
$$= df(x_0)(\upsilon.\Delta t + \alpha(\Delta t)\Delta t)$$
$$+\beta(\upsilon.\Delta t + \alpha(\Delta t)\Delta t).|\upsilon\Delta t + \alpha(\Delta t)\Delta t|$$
$$= df(x_0)(\upsilon).\Delta t + \gamma(\Delta t)\Delta t$$

For a certain $\gamma(\Delta t)$ such that $\gamma(\Delta t) \to 0$ when $\Delta t \to 0$ (we used the linearity of $df(x_0)$). By the definition, this means that the derivative of $f(x(t))$ at $t = t_0$ is exactly $df(x_0)(\upsilon)$. The statement of the theorem can be expressed by a simple formula:

$$\frac{df(x(t))}{dt} = \frac{\partial f}{\partial x^1} x^1 + \ldots + \frac{\partial f}{\partial x^n} x^n \qquad (2)$$

To calculate the value Of df at a point x_0 on a given vector υ one can take an arbitrary curve passing Through x_0 at t_0 with υ as the velocity vector at t_0 and calculate the usual derivative of $f(x(t))$ at $t = t_0$.

Theorem 1.8. For functions $f, g : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$,

$$d(f+g) = df + dg \qquad (1)$$
$$d(fg) = df.g + f.dg \qquad (2)$$

Proof. Consider an arbitrary point x_0 and an arbitrary vector υ stretching from it. Let a curve $x(t)$ be such that $x(t_0) = x_0$ and $x(t_0) = \upsilon$. Hence

$$d(f+g)(x_0)(\upsilon) = \frac{d}{dt}(f(x(t)) + g(x(t)))$$

at $t = t_0$ and

$$d(fg)(x_0)(\upsilon) = \frac{d}{dt}(f(x(t))g(x(t)))$$

at $t = t_0$ Formulae (1) and (2) then immediately follow from the corresponding formulae for the usual derivative Now, almost without change the theory generalizes to functions taking values in $\mathbb{R}^m$ instead of $\mathbb{R}$. The only difference is that now the differential of a map $F : U \to \mathbb{R}^m$ at a point x will be a linear function taking vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$ (instead of $\mathbb{R}$) . For an arbitrary vector $h \in| \mathbb{R}^n$,

$$F(x+h) = F(x) + dF(x)(h)$$
$$+\beta(h)|h| \qquad (3)$$

Where $\beta(h) \to 0$ when $h \to 0$. We have $dF = (dF^1, \ldots, dF^m)$ and

$$dF = \frac{\partial F}{\partial x^1} dx^1 + \ldots + \frac{\partial F}{\partial x^n} dx^n$$

$$= \begin{pmatrix} \frac{\partial F^1}{\partial x^1} \ldots\ldots \frac{\partial F^1}{\partial x^n} \\ \ldots \quad \ldots \quad \ldots \\ \frac{\partial F^m}{\partial x^1} \ldots \frac{\partial F^m}{\partial x^n} \end{pmatrix} \begin{pmatrix} dx^1 \\ \ldots \\ dx^n \end{pmatrix} \qquad (4)$$

In this matrix notation we have to write vectors as vector-columns.

Theorem 1.9. For an arbitrary parametrized curve $x(t)$ in $\mathbb{R}^n$, the differential of a map $F : U \to \mathbb{R}^m$ (where $U \subset \mathbb{R}^n$) maps the velocity vector $x(t)$ to the velocity vector of the curve $F(x(t))$ in $\mathbb{R}^m$:

$$\frac{dF(x(t))}{dt} = dF(x(t))(\dot{x}(t)) \qquad (1)$$

Proof. By the definition of the velocity vector,

$$x(t + \Delta t) = x(t) + \dot{x}(t).\Delta t + \alpha(\Delta t)\Delta t \qquad (2)$$

Where $\alpha(\Delta t) \to 0$ when $\Delta t \to 0$. By the definition of the differential,

$$F(x+h) = F(x) + dF(x)(h) + \beta(h)|h \qquad (3)|$$

Where $\beta(h) \to 0$ when $h \to 0$. we obtain

$$F(x(t+\Delta t)) = F(x + \underbrace{\dot{x}(t).\Delta t + \alpha(\Delta t)\Delta t}_{h})$$

$$= F(x) + dF(x)(\dot{x}(t)\Delta t + \alpha(\Delta t)\Delta t) +$$

$$\beta(\dot{x}(t)\Delta t + \alpha(\Delta t)\Delta t).\left|\dot{x}(t)\Delta t + \alpha(\Delta t)\Delta t\right|$$

$$= F(x) + dF(x)(\dot{x}(t)\Delta t + \gamma(\Delta t)\Delta t$$

For some $\gamma(\Delta t) \to 0$ when $\Delta t \to 0$. This precisely means that $dF(x)\dot{x}(t)$ is the velocity vector of $F(x)$. As every vector attached to a point can be viewed as the velocity vector of some curve passing through this point, this theorem gives a clear geometric picture of dF as a linear map on vectors.

Theorem 1.10 Suppose we have two maps $F : U \to V$ and $G : V \to W$, where $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m, W \subset \mathbb{R}^p$ (open domains). Let $F : x \mapsto y = F(x)$. Then the differential of the composite map $GoF : U \to W$ is the composition of the differentials of F and G:

$$d(GoF)(x) = dG(y)odF(x) \qquad (4)$$

Proof. We can use the description of the differential .Consider a curve $x(t)$ in $\mathbb{R}^n$ with the velocity vector $\dot{x}$. Basically, we need to know to which vector in $\mathbb{R}^p$ it is taken by $d(GoF)$. the curve $(GoF)(x(t) = G(F(x(t))$. By the same theorem, it equals the image under dG of the Anycast Flow vector to the curve $F(x(t))$ in $\mathbb{R}^m$. Applying the theorem once again, we see that the velocity vector to the curve $F(x(t))$ is the image under dF of the vector $\dot{x}(t)$. Hence $d(GoF)(\dot{x}) = dG(dF(\dot{x}))$ for an arbitrary vector $\dot{x}$.

Corollary 1.0. If we denote coordinates in $\mathbb{R}^n$ by $(x^1,...,x^n)$ and in $\mathbb{R}^m$ by $(y^1,...,y^m)$, and write

$$dF = \frac{\partial F}{\partial x^1}dx^1 + ... + \frac{\partial F}{\partial x^n}dx^n \qquad (1)$$

$$dG = \frac{\partial G}{\partial y^1}dy^1 + ... + \frac{\partial G}{\partial y^n}dy^n, \qquad (2)$$

Then the chain rule can be expressed as follows:

$$d(GoF) = \frac{\partial G}{\partial y^1}dF^1 + ... + \frac{\partial G}{\partial y^m}dF^m, \qquad (3)$$

Where dF^i are taken from (1). In other words, to get $d(GoF)$ we have to substitute into (2) the expression for $dy^i = dF^i$ from (3). This can also be expressed by the following matrix formula:

$$d(GoF) = \begin{pmatrix} \frac{\partial G^1}{\partial y^1} \cdots \frac{\partial G^1}{\partial y^m} \\ \ldots \quad \ldots \quad \ldots \\ \frac{\partial G^p}{\partial y^1} \cdots \frac{\partial G^p}{\partial y^m} \end{pmatrix} \begin{pmatrix} \frac{\partial F^1}{\partial x^1} \cdots \frac{\partial F^1}{\partial x^n} \\ \ldots \quad \ldots \quad \ldots \\ \frac{\partial F^m}{\partial x^1} \cdots \frac{\partial F^m}{\partial x^n} \end{pmatrix} \begin{pmatrix} dx^1 \\ \ldots \\ dx^n \end{pmatrix} \qquad (4)$$

i.e., if dG and dF are expressed by matrices of partial derivatives, then $d(GoF)$ is expressed by the product of these matrices. This is often written as

$$\begin{pmatrix} \frac{\partial z^1}{\partial x^1} \cdots \frac{\partial z^1}{\partial x^n} \\ \ldots \quad \ldots \quad \ldots \\ \frac{\partial z^p}{\partial x^1} \cdots \frac{\partial z^p}{\partial x^n} \end{pmatrix} = \begin{pmatrix} \frac{\partial z^1}{\partial y^1} \cdots \frac{\partial z^1}{\partial y^m} \\ \ldots \quad \ldots \quad \ldots \\ \frac{\partial z^p}{\partial y^1} \cdots \frac{\partial z^p}{\partial y^m} \end{pmatrix} \begin{pmatrix} \frac{\partial y^1}{\partial x^1} \cdots \frac{\partial y^1}{\partial x^n} \\ \ldots \quad \ldots \quad \ldots \\ \frac{\partial y^m}{\partial x^1} \cdots \frac{\partial y^m}{\partial x^n} \end{pmatrix}, \qquad (5)$$

Or

$$\frac{\partial z^\mu}{\partial x^a} = \sum_{i=1}^{m} \frac{\partial z^\mu}{\partial y^i}\frac{\partial y^i}{\partial x^a}, \qquad (6)$$

Where it is assumed that the dependence of $y \in \mathbb{R}^m$ on $x \in \mathbb{R}^n$ is given by the map F, the dependence of $z \in \mathbb{R}^p$ on $y \in \mathbb{R}^m$ is given by the map G, and the dependence of $z \in \mathbb{R}^p$ on $x \in \mathbb{R}^n$ is given by the composition GoF.

Definition 1.6. Consider an open domain $U \subset \mathbb{R}^n$. Consider also another copy of $\mathbb{R}^n$, denoted for distinction $\mathbb{R}^n_y$, with the standard coordinates $(y^1...y^n)$. A system of coordinates in the open domain U is given by a map $F : V \to U$, where

$V \subset \mathbb{R}^n_y$ is an open domain of $\mathbb{R}^n_y$, such that the following three conditions are satisfied :

(1) F is smooth;

(2) F is invertible;

(3) $F^{-1} : U \to V$ is also smooth

The coordinates of a point $x \in U$ in this system are the standard coordinates of $F^{-1}(x) \in \mathbb{R}^n_y$

In other words,

$$F : (y^1 ..., y^n) \mapsto x = x(y^1 ..., y^n) \qquad (1)$$

Here the variables $(y^1 ..., y^n)$ are the "new" coordinates of the point x

Example 1.2. Consider a curve in $\mathbb{R}^2$ specified in polar coordinates as

$$x(t) : r = r(t), \varphi = \varphi(t) \qquad (1)$$

We can simply use the chain rule. The map $t \mapsto x(t)$ can be considered as the composition of the maps $t \mapsto (r(t), \varphi(t)), (r, \varphi) \mapsto x(r, \varphi)$. Then, by the chain rule, we have

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial x}{\partial \varphi}\frac{d\varphi}{dt} = \frac{\partial x}{\partial r}\dot{r} + \frac{\partial x}{\partial \varphi}\dot{\varphi} \qquad (2)$$

Here $\dot{r}$ and $\dot{\varphi}$ are scalar coefficients depending on t, whence the partial derivatives $\partial x/\partial r, \partial x/\partial \varphi$ are vectors depending on point in $\mathbb{R}^2$. We can compare this with the formula in the "standard" coordinates: $\dot{x} = e_1 \dot{x} + e_2 \dot{y}$. Consider the vectors $\partial x/\partial r, \partial x/\partial \varphi$. Explicitly we have

$$\frac{\partial x}{\partial r} = (\cos\varphi, \sin\varphi) \qquad (3)$$

$$\frac{\partial x}{\partial \varphi} = (-r\sin\varphi, r\cos\varphi) \qquad (4)$$

From where it follows that these vectors make a basis at all points except for the origin (where $r = 0$). It is instructive to sketch a picture, drawing vectors corresponding to a point as starting from that point. Notice that $\partial x/\partial r, \partial x/\partial \varphi$ are, respectively, the velocity vectors for the curves $r \mapsto x(r, \varphi)$ $(\varphi = \varphi_0 \ fixed)$ and $\varphi \mapsto x(r, \varphi)$ $(r = r_0 \ fixed)$. We can conclude that for an arbitrary curve given in polar coordinates the velocity vector will have components $(\dot{r}, \dot{\varphi})$ if as a basis we take $e_r := \partial x/\partial r, e_\varphi := \partial x/\partial \varphi$:

$$\dot{x} = e_r \dot{r} + e_\varphi \dot{\varphi} \qquad (5)$$

A characteristic feature of the basis e_r, e_φ is that it is not "constant" but depends on point. Vectors "stuck to points" when we consider curvilinear coordinates.

Proposition 1.3. The velocity vector has the same appearance in all coordinate systems.

Proof. Follows directly from the chain rule and the transformation law for the basis e_i .In particular, the elements of the basis $e_i = \partial x/\partial x^i$ (originally, a formal notation) can be understood directly as the velocity vectors of the coordinate lines $x^i \mapsto x(x^1, ..., x^n)$ (all coordinates but x^i are fixed). Since we now know how to handle velocities in arbitrary coordinates, the best way to treat the differential of a map $F : \mathbb{R}^n \to \mathbb{R}^m$ is by its action on the velocity vectors. By definition, we set

$$dF(x_0) : \frac{dx(t)}{dt}(t_0) \mapsto \frac{dF(x(t))}{dt}(t_0) \qquad (1)$$

Now $dF(x_0)$ is a linear map that takes vectors attached to a point $x_0 \in \mathbb{R}^n$ to vectors attached to the point $F(x) \in \mathbb{R}^m$

$$dF = \frac{\partial F}{\partial x^1}dx^1 + ... + \frac{\partial F}{\partial x^n}dx^n$$

$$(e_1, ..., e_m)\begin{pmatrix} \frac{\partial F^1}{\partial x^1} \cdots \frac{\partial F^1}{\partial x^n} \\ \cdots \ \cdots \ \cdots \\ \frac{\partial F^m}{\partial x^1} \cdots \frac{\partial F^m}{\partial x^n} \end{pmatrix}\begin{pmatrix} dx^1 \\ \cdots \\ dx^n \end{pmatrix}, \qquad (2)$$

In particular, for the differential of a function we always have

$$df = \frac{\partial f}{\partial x^1}dx^1 + ... + \frac{\partial f}{\partial x^n}dx^n, \qquad (3)$$

Where x^i are arbitrary coordinates. The form of the differential does not change when we perform a change of coordinates.

Example 1.3 Consider a 1-form in $\mathbb{R}^2$ given in the standard coordinates:

$A=-ydx+xdy$ In the polar coordinates we will have $x=r\cos\varphi, y=r\sin\varphi$, hence

$$dx=\cos\varphi dr-r\sin\varphi d\varphi$$

$$dy=\sin\varphi dr+r\cos\varphi d\varphi$$

Substituting into A, we get

$$A=-r\sin\varphi(\cos\varphi dr-r\sin\varphi d\varphi)$$
$$+r\cos\varphi(\sin\varphi dr+r\cos\varphi d\varphi)$$
$$=r^2(\sin^2\varphi+\cos^2\varphi)d\varphi=r^2d\varphi$$

Hence $A=r^2d\varphi$ is the formula for A in the polar coordinates. In particular, we see that this is again a 1-form, a linear combination of the differentials of coordinates with functions as coefficients. Secondly, in a more conceptual way, we can define a 1-form in a domain U as a linear function on vectors at every point of U : $\omega(\upsilon)=\omega_1\upsilon^1+...+\omega_n\upsilon^n$, (1)

If $\upsilon=\sum e_i\upsilon^i$, where $e_i=\partial x/\partial x^i$. Recall that the differentials of functions were defined as linear functions on vectors (at every point), and $dx^i(e_j)=dx^i\left(\frac{\partial x}{\partial x^j}\right)=\delta^i_j$ (2) at every point x.

Theorem 1.9. For arbitrary 1-form ω and path γ, the integral $\int_\gamma\omega$ does not change if we change parametrization of γ provide the orientation remains the same.

Proof: Consider $\left\langle\omega(x(t)),\frac{dx}{dt}\right\rangle$ and $\left\langle\omega(x(t(t'))),\frac{dx}{dt'}\right\rangle$ As

$$\left\langle\omega(x(t(t'))),\frac{dx}{dt'}\right\rangle=\left|\left\langle\omega(x(t(t'))),\frac{dx}{dt'}\right\rangle\right|.\frac{dt}{dt'},$$

Let p be a rational prime and let $K=\mathbb{Q}(\zeta_p)$. We write ζ for ζ_p or this section. Recall that K has degree $\varphi(p)=p-1$ over $\mathbb{Q}$. We wish to show that $O_K=\mathbb{Z}[\zeta]$. Note that ζ is a root of x^p-1, and thus is an algebraic integer; since O_K is a ring we have that $\mathbb{Z}[\zeta]\subseteq O_K$. We give a proof without assuming unique factorization of ideals. We begin with some norm and trace computations. Let j be an integer. If j is not divisible by p, then ζ^j is a primitive p^{th} root of unity, and thus its conjugates are $\zeta,\zeta^2,...,\zeta^{p-1}$. Therefore

$$Tr_{K/\mathbb{Q}}(\zeta^j)=\zeta+\zeta^2+...+\zeta^{p-1}=\Phi_p(\zeta)-1=-1$$

If p does divide j, then $\zeta^j=1$, so it has only the one conjugate 1, and $Tr_{K/\mathbb{Q}}(\zeta^j)=p-1$ By linearity of the trace, we find that

$$Tr_{K/\mathbb{Q}}(1-\zeta)=Tr_{K/\mathbb{Q}}(1-\zeta^2)=...$$
$$=Tr_{K/\mathbb{Q}}(1-\zeta^{p-1})=p$$

We also need to compute the norm of $1-\zeta$. For this, we use the factorization

$$x^{p-1}+x^{p-2}+...+1=\Phi_p(x)$$
$$=(x-\zeta)(x-\zeta^2)...(x-\zeta^{p-1});$$

Plugging in $x=1$ shows that

$$p=(1-\zeta)(1-\zeta^2)...(1-\zeta^{p-1})$$

Since the $(1-\zeta^j)$ are the conjugates of $(1-\zeta)$, this shows that $N_{K/\mathbb{Q}}(1-\zeta)=p$ The key result for determining the ring of integers O_K is the following.

LEMMA 1.9

$$(1-\zeta)O_K\cap\mathbb{Z}=p\mathbb{Z}$$

Proof. We saw above that p is a multiple of $(1-\zeta)$ in O_K, so the inclusion $(1-\zeta)O_K\cap\mathbb{Z}\supseteq p\mathbb{Z}$ is immediate. Suppose now that the inclusion is strict. Since $(1-\zeta)O_K\cap\mathbb{Z}$ is an ideal of $\mathbb{Z}$ containing $p\mathbb{Z}$ and $p\mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$, we must have $(1-\zeta)O_K\cap\mathbb{Z}=\mathbb{Z}$ Thus we can write

$$1=\alpha(1-\zeta)$$

For some $\alpha\in O_K$. That is, $1-\zeta$ is a unit in O_K.

COROLLARY 1.1 For any $\alpha\in O_K$, $Tr_{K/\mathbb{Q}}((1-\zeta)\alpha)\in p\mathbb{Z}$

PROOF. We have

$$Tr_{K/\mathbb{Q}}((1-\zeta)\alpha)=\sigma_1((1-\zeta)\alpha)+...+\sigma_{p-1}((1-\zeta)\alpha)$$
$$=\sigma_1(1-\zeta)\sigma_1(\alpha)+...+\sigma_{p-1}(1-\zeta)\sigma_{p-1}(\alpha)$$
$$=(1-\zeta)\sigma_1(\alpha)+...+(1-\zeta^{p-1})\sigma_{p-1}(\alpha)$$

Where the σ_i are the complex embeddings of K (which we are really viewing as automorphisms of K) with the usual ordering. Furthermore, $1-\zeta^j$ is a multiple of $1-\zeta$ in O_K for every $j \neq 0$. Thus $Tr_{K/\mathbb{Q}}(\alpha(1-\zeta)) \in (1-\zeta)O_K$ Since the trace is also a rational integer.

PROPOSITION 1.4 Let p be a prime number and let $K = \mathbb{Q}(\zeta_p)$ be the p^{th} cyclotomic field. Then $O_K = \mathbb{Z}[\zeta_p] \cong \mathbb{Z}[x]/(\Phi_p(x))$; Thus $1, \zeta_p, ..., \zeta_p^{p-2}$ is an integral basis for O_K.

PROOF. Let $\alpha \in O_K$ and write $\alpha = a_0 + a_1\zeta + ... + a_{p-2}\zeta^{p-2}$ With $a_i \in \mathbb{Q}$. Then

$$\alpha(1-\zeta) = a_0(1-\zeta) + a_1(\zeta-\zeta^2) + ... + a_{p-2}(\zeta^{p-2} - \zeta^{p-1})$$

By the linearity of the trace and our above calculations we find that $Tr_{K/\mathbb{Q}}(\alpha(1-\zeta)) = pa_0$ We also have $Tr_{K/\mathbb{Q}}(\alpha(1-\zeta)) \in p\mathbb{Z}$, so $a_0 \in \mathbb{Z}$ Next consider the algebraic integer $(\alpha - a_0)\zeta^{-1} = a_1 + a_2\zeta + ... + a_{p-2}\zeta^{p-3}$; This is an algebraic integer since $\zeta^{-1} = \zeta^{p-1}$ is. The same argument as above shows that $a_1 \in \mathbb{Z}$, and continuing in this way we find that all of the a_i are in $\mathbb{Z}$. This completes the proof.

Example 1.4 Let $K = \mathbb{Q}$, then the local ring $\mathbb{Z}_{(p)}$ is simply the subring of $\mathbb{Q}$ of rational numbers with denominator relatively prime to p. Note that this ring $\mathbb{Z}_{(p)}$ is not the ring $\mathbb{Z}_p$ of p-adic integers; to get $\mathbb{Z}_p$ one must complete $\mathbb{Z}_{(p)}$. The usefulness of $O_{K,p}$ comes from the fact that it has a particularly simple ideal structure. Let a be any proper ideal of $O_{K,p}$ and consider the ideal $a \cap O_K$ of O_K. We claim that $a = (a \cap O_K)O_{K,p}$; That is, that a is generated by the elements of a in $a \cap O_K$. It is clear from the definition of an ideal that $a \supseteq (a \cap O_K)O_{K,p}$. To prove the other inclusion, let α be any element of a. Then we can write $\alpha = \beta/\gamma$ where $\beta \in O_K$ and $\gamma \notin p$. In particular, $\beta \in a$ (since $\beta/\gamma \in a$ and a is an ideal), so $\beta \in O_K$ and $\gamma \notin p$. so $\beta \in a \cap O_K$. Since $1/\gamma \in O_{K,p}$, this implies that $\alpha = \beta/\gamma \in (a \cap O_K)O_{K,p}$, as claimed.We can use this fact to determine all of the ideals of $O_{K,p}$. Let a be any ideal of $O_{K,p}$ and consider the ideal factorization of $a \cap O_K$ in O_K. write it as $a \cap O_K = p^n b$ For some n and some ideal b, relatively prime to p. we claim first that $bO_{K,p} = O_{K,p}$. We now find that

$$a = (a \cap O_K)O_{K,p} = p^n bO_{K,p} = p^n O_{K,p}$$

Since $bO_{K,p}$. Thus every ideal of $O_{K,p}$ has the form $p^n O_{K,p}$ for some n; it follows immediately that $O_{K,p}$ is noetherian. It is also now clear that $p^n O_{K,p}$ is the unique non-zero prime ideal in $O_{K,p}$. Furthermore, the inclusion $O_K \mapsto O_{K,p}/pO_{K,p}$ Since $pO_{K,p} \cap O_K = p$, this map is also surjection, since the residue class of $\alpha/\beta \in O_{K,p}$ (with $\alpha \in O_K$ and $\beta \notin p$) is the image of $\alpha\beta^{-1}$ in $O_{K/p}$, which makes sense since β is invertible in $O_{K/p}$. Thus the map is an isomorphism. In particular, it is now abundantly clear that every non-zero prime ideal of $O_{K,p}$ is maximal. To show that $O_{K,p}$ is a Dedekind domain, it remains to show that it is integrally closed in K. So let $\gamma \in K$ be a root of a polynomial with coefficients in $O_{K,p}$; write this polynomial as $x^m + \frac{\alpha_{m-1}}{\beta_{m-1}}x^{m-1} + ... + \frac{\alpha_0}{\beta_0}$ With $\alpha_i \in O_K$ and $\beta_i \in O_{K-p}$. Set $\beta = \beta_0\beta_1...\beta_{m-1}$. Multiplying by β^m we find that $\beta\gamma$ is the root of a monic polynomial with coefficients in O_K. Thus $\beta\gamma \in O_K$; since $\beta \notin p$, we have $\beta\gamma/\beta = \gamma \in O_{K,p}$. Thus $O_{K,p}$ is integrally close in K.

COROLLARY 1.2. Let K be a number field of degree n and let α be in O_K then

$$N'_{K/\mathbb{Q}}(\alpha O_K) = \left|N_{K/\mathbb{Q}}(\alpha)\right|$$

PROOF. We assume a bit more Galois theory than usual for this proof. Assume first that $K/\mathbb{Q}$ is Galois. Let σ be an element of $Gal(K/\mathbb{Q})$. It is clear that $\sigma(O_K)/\sigma(\alpha) \cong O_{K/\alpha}$; since $\sigma(O_K) = O_K$, this shows that $N'_{K/\mathbb{Q}}(\sigma(\alpha)O_K) = N'_{K/\mathbb{Q}}(\alpha O_K)$. Taking the product over all $\sigma \in Gal(K/\mathbb{Q})$, we have $N'_{K/\mathbb{Q}}(N_{K/\mathbb{Q}}(\alpha)O_K) = N'_{K/\mathbb{Q}}(\alpha O_K)^n$ Since $N_{K/\mathbb{Q}}(\alpha)$ is a rational integer and O_K is a free $\mathbb{Z}$ -module of rank n,

$O_K / N_{K/\mathbb{Q}}(\alpha)O_K$ Will have order $N_{K/\mathbb{Q}}(\alpha)^n$; therefore

$$N'_{K/\mathbb{Q}}(N_{K/\mathbb{Q}}(\alpha)O_K) = N_{K/\mathbb{Q}}(\alpha O_K)^n$$

This completes the proof. In the general case, let L be the Galois closure of K and set $[L:K]=m.$

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REFERENCES

[1] Dynamics and Control of Large Electric Power Systems. Ilic, M. and Zaborszky, J. John Wiley & Sons, Inc. © 2000, p. 756.

[2] Modeling and Evaluation of Intrusion Tolerant Systems Based on Dynamic Diversity Backups. Meng, K. et al. Proceedings of the 2009 International Symposium on Information Processing (ISIP'09). Huangshan, P. R. China, August 21-23, 2009, pp. 101–104

[3] Characterizing Intrusion Tolerant Systems Using A State Transition Model. Gong, F. et al., April 24, 2010.

[4] Energy Assurance Daily, September 27, 2007. U.S. Department of Energy, Office of Electricity Delivery and Energy Reliability, Infrastructure Security and Energy Restoration Division. April 25, 2010.

[5] CENTIBOTS Large Scale Robot Teams. Konoledge, Kurt et al. Artificial Intelligence Center, SRI International, Menlo Park, CA 2003.

[6] Handling Communication Restrictions and Team Formation in Congestion Games, Agogino, A. and Tumer, K. Journal of Autonomous Agents and Multi Agent Systems, 13(1):97–115, 2006.

[7] Robotics and Autonomous Systems Research, School of Mechanical, Industrial and Manufacturing Engineering, College of Engineering, Oregon State University

[8] D. Dietrich, D. Bruckner, G. Zucker, and P. Palensky, "Communication and computation in buildings: A short introduction and overview," *IEEE Trans. Ind. Electron.*, vol. 57, no. 11, pp. 3577–3584, Nov. 2010.

[9] V. C. Gungor and F. C. Lambert, "A survey on communication networks for electric system automation," *Comput. Networks*, vol. 50, pp. 877–897, May 2006.

[10] S. Paudyal, C. Canizares, and K. Bhattacharya, "Optimal operation of distribution feeders in smart grids," *IEEE Trans. Ind. Electron.*, vol. 58, no. 10, pp. 4495–4503, Oct. 2011.

[11] D. M. Laverty, D. J. Morrow, R. Best, and P. A. Crossley, "Telecommunications for smart grid: Backhaul solutions for the distribution network," in *Proc. IEEE Power and Energy Society General Meeting*, Jul. 25–29, 2010, pp. 1–6.

[12] L. Wenpeng, D. Sharp, and S. Lancashire, "Smart grid communication network capacity planning for power utilities," in *Proc. IEEE PES, Transmission Distrib. Conf. Expo.*, Apr. 19–22, 2010, pp. 1–4.

[13] Y. Peizhong, A. Iwayemi, and C. Zhou, "Developing ZigBee deployment guideline under WiFi interference for smart grid applications," *IEEE Trans. Smart Grid*, vol. 2, no. 1, pp. 110–120, Mar. 2011.

[14] C. Gezer and C. Buratti, "A ZigBee smart energy implementation for energy efficient buildings," in *Proc. IEEE 73rd Veh. Technol. Conf. (VTC Spring)*, May 15–18, 2011, pp. 1–5.

[15] R. P. Lewis, P. Igic, and Z. Zhongfu, "Assessment of communication methods for smart electricity metering in the U.K.," in *Proc. IEEE PES/IAS Conf. Sustainable Alternative Energy (SAE)*, Sep. 2009, pp. 1–4.

[16] A. Yarali, "Wireless mesh networking technology for commercial and industrial customers," in *Proc. Elect. Comput. Eng., CCECE*,May 1–4, 2008, pp. 000047–000052.

[17] M. Y. Zhai, "Transmission characteristics of low-voltage distribution networks in China under the smart grids environment," *IEEE Trans. Power Delivery*, vol. 26, no. 1, pp. 173–180, Jan. 2011.

[18] V. Paruchuri, A. Durresi, and M. Ramesh, "Securing powerline communications," in *Proc. IEEE Int. Symp. Power Line Commun. Appl., (ISPLC)*, Apr. 2–4, 2008, pp. 64–69.

[19] Q.Yang, J. A. Barria, and T. C. Green, "Communication infrastructures for distributed control of power distribution networks," *IEEE Trans. Ind. Inform.*, vol. 7, no. 2, pp. 316–327, May 2011.

[20] T. Sauter and M. Lobashov, "End-to-end communication architecture for smart grids," *IEEE Trans. Ind. Electron.*, vol. 58, no. 4, pp. 1218–1228, Apr. 2011.

[21] K. Moslehi and R. Kumar, "Smart grid—A reliability perspective," *Innovative Smart Grid Technologies (ISGT)*, pp. 1–8, Jan. 19–21, 2010.

[22] Southern Company Services, Inc., "Comments request for information on smart grid communications requirements," Jul. 2010

[23] R. Bo and F. Li, "Probabilistic LMP forecasting considering load uncertainty," *IEEE Trans. Power Syst.*, vol. 24, pp. 1279–1289, Aug. 2009.

[24] *Power Line Communications*, H. Ferreira, L. Lampe, J. Newbury, and T. Swart (Editors), Eds. New York: Wiley, 2010.

[25] G. Bumiller, "Single frequency network technology for fast ad hoc communication networks over power lines," WiKu-Wissenschaftsverlag Dr. Stein 2010.

[31] G. Bumiller, L. Lampe, and H. Hrasnica, "Power line communications for large-scale control and automation systems," *IEEE Commun. Mag.*, vol. 48, no. 4, pp. 106–113, Apr. 2010.

[32] M. Biagi and L. Lampe, "Location assisted routing techniques for power line communication in smart grids," in *Proc. IEEE Int. Conf. Smart Grid Commun.*, 2010, pp. 274–278.

[33] J. Sanchez, P. Ruiz, and R. Marin-Perez, "Beacon-less geographic routing made partical: Challenges, design guidelines and protocols," *IEEE Commun. Mag.*, vol. 47, no. 8, pp. 85–91, Aug. 2009.

[34] N. Bressan, L. Bazzaco, N. Bui, P. Casari, L. Vangelista, and M. Zorzi, "The deployment of a smart monitoring system using wireless sensors and actuators networks," in *Proc. IEEE Int. Conf. Smart Grid Commun. (SmartGridComm)*, 2010, pp. 49–54.

[35] S. Dawson-Haggerty, A. Tavakoli, and D. Culler, "Hydro: A hybrid routing protocol for low-power and lossy networks," in *Proc. IEEE Int. Conf. Smart Grid Commun. (SmartGridComm)*, 2010, pp. 268–273.

[36] S. Goldfisher and S. J. Tanabe, "IEEE 1901 access system: An overview of its uniqueness and motivation," *IEEE Commun. Mag.*, vol. 48, no. 10, pp. 150–157, Oct. 2010.

[37] V. C. Gungor, D. Sahin, T. Kocak, and S. Ergüt, "Smart grid communications and networking," Türk Telekom, Tech. Rep. 11316-01, Apr 2011.